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STUDENTS' FUNCTIONAL REPRESENTATIONS AND CONCEPTIONS IN THE CONSTRUCTION OF MATHEMATICAL CONCEPTS. AN EXAMPLE: THE CONCEPT OF LIMIT

Abstract. The role of mental representations and imagery has been studied for several years in order to explain the processes of constructing concepts and to elucidate the mathematical abilities of students. Searching for new ways for the construction of mathematical concepts and problem solving strategies, the Working group on Representations and Mathematics Visualization of PME-NA, 1998-2002 (see Hitt, 2002) highlighted the importance of semiotic representations while building mathematical concepts, giving a new dimension to research work in mathematics education. Taking into account previous research done by Duval (1993, 1995, 1999) on constructing mathematical concepts, we focused on students' conceptions and on the role of the functional representations (spontaneous representations) used by the students in order to build a mathematical concept. We found that the representations used by the students when constructing a mathematical concept play a significant role and that they are part of their conception. These functional representations are of a kind usually not found nor used by mathematics teachers.

Résumé. Représentations fonctionnelles et conceptions dans la construction de concepts mathématiques. Un exemple : Le concept de la limite.
Le rôle des représentations mentales et leur manipulation a été étudié pendant plusieurs années pour expliquer les processus de construction des concepts et pour comprendre les capacités mathématiques des étudiants. Recherchant de nouvelles voies sur la construction des concepts et des stratégies mathématiques sur la résolution des problèmes, le groupe de travail « Representations and mathematics visualization » du PME-NA, 1998-2002 (voir Hitt, 2002) a mis en valeur l'importance des représentations sémiotiques sur les constructions des concepts mathématiques, donnant une nouvelle dimension de travail de recherches dans la didactique des mathématiques. Tenant compte de la recherche précédente faite par Duval (1993, 1995, 1999) sur la construction des concepts mathématiques, nous nous sommes concentrés sur les conceptions des étudiants et sur le rôle des représentations fonctionnelles (représentations spontanées) employées par les étudiants afin de construire un concept mathématique. Nous avons constaté que ces représentations employées par les étudiants en construisant un concept jouent un rôle significatif et sont une partie de leur conception. Ces représentations fonctionnelles sont un genre de représentations qui diffèrent habituellement de ceux que nous trouvons dans les manuels, ou ceux qu'utilisent les professeurs dans la classe de mathématiques.

Mots-clés. Représentations fonctionnelles, conceptions et registres sémiotiques.

ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES, volume 11 p. 251 – 267.

1. Introduction

What we present here has to do with spontaneous semiotic representations, registers of representations, and construction of mathematical concepts. Our research work stresses the functional character of the representations in order to understand their role in the process of learning. Hence, we direct our attention to the whole of the semiotic representations produced by students; for instance, we focus on the building of the concept of limit.

Going back to the 1960's, we find a kind of a psychological approach to this theme. For example, Guilford (1967) in his book "The nature of human intelligence", used a questionnaire to detect 120 abilities in humans and came up with his three-dimensional model. He was aware of isolating abilities and trying to find correlations among them. In the 1970's, different approaches in mathematics education to explain mathematics abilities in schoolchildren appeared. Krutetskii (1976), for instance, documented performances of the school children when facing mathematical problems, related to "inspiration" or "insight" (Idem, p. 156) and mental processes (Idem, p. 309). Little by little, the mental representation became a subject of study in mathematics education.

In the 1980's, a new approach, in mathematics education surfaced; Tall and Vinner (1981, p. 151), Vinner (1983, 1994) and Tall (1991) give certain precisions to clarify the mental representations. For example, Tall (Idem) gives the definition of concept image:

> "We shall use the term concept image to describe the total cognitive structure that is associated with the concepts, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures…" *(Tall, 1991, p. 7)*

This approach is based on the importance of what students think about one concept and on a general idea about the problems the students might experience in the construction of mathematical objects, but that approach seems to be too broad and does not provide enough information about the construction of concepts. A few years later, the need for taking into account the mental representations was emphasized, both in mathematics education and psychology (see for example, Richard, 1990/1998).

Duroux (cited by Brousseau, 1997, 99-100) related to conceptions and epistemological obstacle, writes:

a) An obstacle is a piece of knowledge or a conception, not a difficulty or a lack of knowledge;

b) This piece of knowledge produces responses which are appropriate within a particular, frequently experienced, context;

c) But it generates false responses outside this context. A correct, universal response requires a notably different point of view;

d) Finally, this piece of knowledge withstands both occasional contradictions and the establishment of a better piece of knowledge. Possession of a better piece of knowledge is not sufficient for the preceding one to disappear (this distinguishes between the overcoming of obstacles and Piaget's adaptation). It is therefore essential to identify it and to incorporate its rejection into the new piece of knowledge;

e) After its inaccuracy has been recognized, it continues to crop up in an untimely, persistent way.

From that point of view, with specific activities, we can detect some epistemological obstacles students' have, and to develop some activities to promote contradictions in students performances and a reflection about their productions to overcome the obstacle. The question is how students have built an obstacle? An approximation to answer this question is to analyse the evolution of a concept in history of mathematics.

Balacheff et Gaudin's (2002, p. 6) approach about conceptions is as follows:

"We call conception C a quadruplet (P, R, L, Σ) in which:

- P is a set of problems;
- R is a set of operators;
- L is a representation system;
- Σ is a control structure.

The question of the concrete characterisation of *the set P of problems* is complex. One option would be to consider all the problems for which the considered conception provides efficient tools to elaborate a solution [Vergnaud approach, 1991]… Another option could consist of considering a finite set of problems with the idea that other problems will derive from them [Brousseau approach, 1997]… we propose to adopt a pragmatic position, deriving the description of P, in an empirical way, from the characterization of situations allowing to diagnose students' conceptions.
…*The set R of operators* is more classical. Operators are means to obtain an evolution of the relation between the subject and the milieu; they are the tools for action…
…*The representation system L* consists of a repertory of structured set of signifiers, of a linguistic nature or not, used at the interface between the subject and the milieu, supporting action and feedback, operations and

> decisions. Just to mention a few examples: algebraic language, geometrical drawing, natural language, but also interfaces of mathematical software and calculators are all examples of representation systems.
>
> …*The control structure* Σ, is constituted by all the means needed in order to make choices, to take decisions, as well as to express judgement. (p. 6-7)

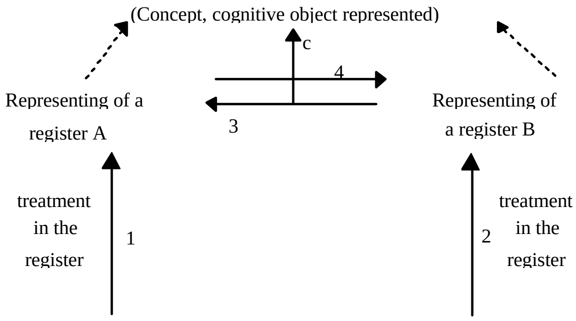
To illustrate their theoretical approach, Balacheff and Gaudin showed some activities with Cabri and used their framework to characterize students' conceptions. From this point of view, implicitly, they are taking into account the institutional representations (Cabri environment) in the constructions of conceptions; as a consequence, in that environment there is little place to understand students' representations and, the students are induced to use the representations that the software allows. We will return to this problem in what follows.

During the 1990's and after the publication of the book "Problems of representation in the teaching and learning of mathematics" (edited by Claude Janvier, 1987), a new approach related to the role of representations in the learning of mathematics appeared. Researchers paid attention to the role of the semiotic representations in building of mathematical concepts taking into account specific activities and reflecting on specific knowledge. For example, Duval (1988, 1993) was interested in the difficulties students have when passing from one representation to another. He found that it is important to clarify the difficulties students encounter when moving from one representation to another and he analysed thoroughly what a representation represents. This led him to introduce the notion of *visual variable*. For instance, in case of a linear function, the question asked is what visual variables we need to consider in order to construct an algebraic representation of a linear function from a geometric representation. He found that there are 18 variables, in general, to associate a graph of a linear function with an algebraic expression. In 1993, Duval gave specific details about the construction of concepts, stressing the fact that a mathematical representation only partially represents the mathematical object in question, and taking into account the main activity of conversion between representations he introduced a new notion, that of register:

A semiotic system may be a representational register if it allows for three cognitive activities associated to semiotics:

1) The presence of an identifiable representation…;
2) The treatment of a representation, which is the transformation of a representation within the same register where it was formed…;

3) The conversion of a representation which is the transformation of the representation into a different register which preserves the totality or part of the meaning of the initial representation…



(Duval, 1993, p. 41).

From this theoretical approach, the analysis of errors focuses on students' performances when converting from one representation to another. When a student is articulating several representations of an object, the student is indeed constructing the mathematical concept. Then, for Duval, the concept is the mathematical idea of an official form of knowledge, shared in an academic community.

Thus, from Duval's point of view, we can explain the existence of some students' errors in terms of a lack of coordination among representations. Implicitly, Duval deals with institutional representations and using them in some activities we can detect the kind of errors students produce when converting from one representation to another.

2. Functional representations and conceptions

The question that arises from this view could be formulated as follows: How do the institutional representations used in the classroom influence students in their knowledge acquisition? And, if the students have not constructed a mathematical concept accepted by the academic community, what kind of knowledge have they constructed?

On one hand, we defined functional representations, as the spontaneous representations that a student uses in a mathematical situation. On the other hand, we named institutional representations, the representations found in books or on computer screens, or those used by teachers when explaining to students on the blackboard.

In this work, a conception is a personal knowledge, constructed by an individual, personally or social in interaction that is not equivalent to the institutionalized knowledge. It is possible to detect a conception of a person trough the spontaneous representations a person uses when solving a mathematical task. Thus, a conception could be:

– an epistemological obstacle;

– a partial construction of a concept, coherent construction of some representations and their conversion from one representation to another;

– partial construction of a concept that works in certain contexts and not in others, but not necessarily represents an epistemological obstacle;

– coherent blend of functional representations.

In this document, we are interested in the last two characterisations; the first two are well described in literature. To characterize the fourth cognitive construction, we need a more elaborated approach and we will do so in the next section. To illustrate the third kind of construction, I would like to show the following example that results from our research in connection with an interview carried out with a high school teacher. We asked her to give us the definition of the derivative of a function. She relied on writing a classical definition as follow:

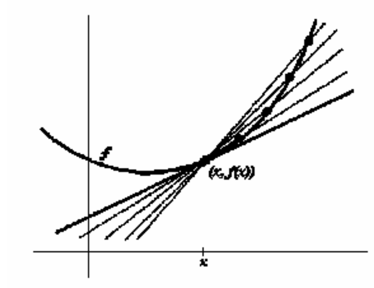
Teacher said: *"A function f is differentiable at one point 'x' if the limit* $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ *exists."* She was asked to explain her definition and, at the blackboard, she draw a graph:	

Figure 1: Student's (high school teacher) definition of differentiability.

We can see that she gave us an algebraic and a graphic representation usually found in textbooks. From this point of view, is not easy to know which kind of knowing she has constructed related to the concept of differentiability. Then, to elucidate about her cognitive construction, she was given the following task:

Given $f(x)=\begin{cases}(x+1)^2 & if \quad x\le 0\\ (x-1)^2 & if \quad x>0\end{cases}$; is the function differentiable at x = 0?

She drew the followings graphs:

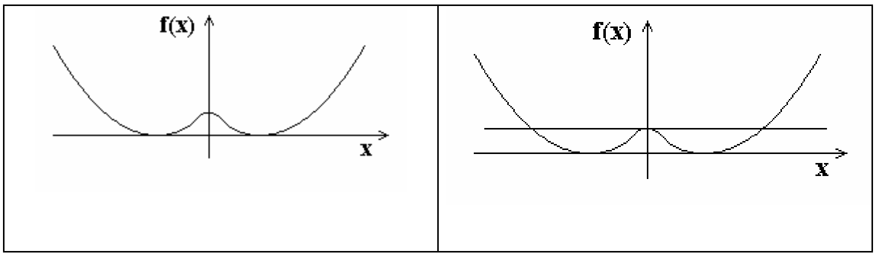


Figure 2: Student's geometrical answer to the above question.

Then, she said that *"…the function is differentiable at x = 0."* She was asked to take her response as a conjecture to give an algebraic answer. Her answer was:

$$\text{“}\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{(h-1)^2-(0+1)^2}{h}=\lim_{h\to 0}\frac{h^2-2h}{h}=\lim_{h\to 0}(h-2)=-2\text{”}$$

As one can see, the teacher took h as positive. The interviewer asked her about the values of h in her definition and she said: "***h is positive***".

Here we can observe her spontaneous representations and can deduct what kind of conception she has constructed; that is, in her definition of differentiable function and in her graphic representation, **h is positive**; a conception that was probably generated by the interaction with the institutional representations of the concept of differentiability of a function used in her process of learning.

At this moment, the interviewer asked her to analyze her graphical answer and her algebraic process to give a definitive answer. She realised that she was in a contradictory situation. Finally she gave the right answer.

Following this approach, the question remains, how do students construct concepts?

We would like to reflect on this point, and look closely at the spontaneous representations students produce in the construction of mathematical concepts.

3. Methodology of our experimentation

In our research project, we designed several activities to be used in a collaborative (Davison, 1998; Dillenbourg, 1999) and scientific debate learning environment (Alibert and Thomas, 1991; Legrand, 2001) and self-reflection (Hadamard, 1975) (reconstruction of the activity as an individual task at home). In our teaching experiment, we used this methodology as a tool for improving the ideas of participants and for transforming false intuitions into consistent knowledge. The participants included twenty-one students (high school teachers) taking a course in a master's degree program in mathematics education. Twenty-two activities were designed (see Hitt & Páez, 2004). The preliminary activities, about finite and infinite processes, drew out participants' intuitive ideas about potential infinity and, in a second phase, activities promoted a conflict between their ideas about potential infinity and actual infinity. Conceptualizing actual infinity was necessary to solve certain mathematical problems related to limits of sequences and series (some convergent and some not).

In the first phase of our research, we used a diagnostic questionnaire about functions, limits and infinity and based on the results of the questionnaire, we constituted small working groups for the entire course. We would like to describe some characteristics of some students during the course.

With the results of the diagnostic questionnaire (13 questions) and the academic profile of each student, we made a classification. We give a sample of answers from some of the students to one question (see Table 1).

Diagnostic questionnaire: Example of classification with one question

Fictitious names	***Question: What is the meaning of*** $\lim_{x \to a} f(x) = L$ ***?***	***Classification***
Juan	*As we approach arbitrarily on a horizontal axes, from left and right a value* $x = a$ *belonging (or not) to domain of a function f(x), the value of the image is approaching a fixed limit L, which belongs or not to the image set of the function.*	*Intuitive*
Lidia	*That means that* $\forall\, \varepsilon > 0\ \exists\, \delta > 0 \ni$ $\lvert x - a \rvert < \delta \Rightarrow \lvert f(x) - f(a) \rvert < \varepsilon$ *or* $\lvert f(x) - L \rvert < \varepsilon$	*Formalist (she gave a definition of continuity)*
Victor	*That means that the value of the function* f *is approaching "L" as we are taking numbers "x" very near to "a", that is: as "x" becomes closer to "a" (by the left and right), then* f *will be closer to "L".*	*Intuitive*
Adrian	*In a neighbourhood of the point "a" (as close as we wish) the function is approaching the value L, and as we approach the point 'a' we can approach as close as we wish the point L.*	*Intuitive*
Pedro	*That means that for every* $\forall\, \varepsilon > 0\ \exists\, \delta > 0 \mid\ \lvert f(x) - L \rvert < \varepsilon \Leftrightarrow \varepsilon > \delta$.	*Formalist and contradictory*

Table 1: Diagnostic questionnaire: Example of classification with one question.

We decided to put together an intuitive person with a formalist and a third one who made contradictory statements in some parts of the questionnaire. Before this course, students had constructed some conceptions (personal constructs) that are not the social knowledge recognized by university professors. In this regard, we paid special attention to the representations students used when solving a mathematical problem in their learning process.

In general, in our teaching method, we tried to generate a cognitive conflict in case the concept of limit was not constructed coherently. That is, when designing the activities, we tried to construct or to use activities already tested in other experiments, to generate a cognitive conflict in case the concept of limit was not constructed coherently. We took into account this point of view, and also the role of semiotic representations, within a co-operative learning and scientific debate and self-reflection activity. Activities in the class were designed to follow this trend (see Hitt 2003; Hitt & Paez 2004 and Hitt & Borbon, 2004).

In this work, our attention focused on how students constructed the concept of limit. Hence, our discussion will also put emphasis on this aspect.

4. Work in groups, scientific debate and individual work

After the third session, students were asked to write a definition of limit as part of their individual work. Victor wrote: “For me, the expression $\lim_{x\to a} f(x) = L$ means that the value of the function f is approaching as close as you wish a value ‘*L*’ taking values closer to ‘a’. This means L is a value that NEVER is reached, but we are very near to it, as near as our mind can imagine. In a practical way, the value ‘*L*’ represents an ideal, something we would like to reach if there were ‘a last’ value…”

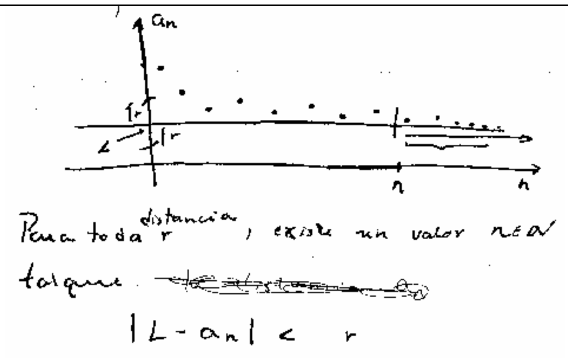
If we compare the two definitions, the initial answer in the diagnostic questionnaire see Table 1) and this one, we can observe that Victor’s intuitive idea has changed, i.e., the idea of “approaching” has changed to “approaching as close as you wish”.

Later on, during the study, students discussed how to construct a definition of a convergent sequence making use of the absolute value (neighbourhood). One of the students suggested that if we have convergence we must have this property $|L - a_{n-1}| > |L - a_n|$ for all n; somebody trying to construct a counter example proposed the following:

$$3, 2, \frac{3}{2}, \frac{2}{2}, \frac{3}{3}, \frac{2}{3}, \frac{3}{4}, \frac{2}{4}, \frac{3}{5}, \frac{2}{5}, \ldots$$

At this moment, the professor asked the students to work again in groups, in order to construct a definition. At this occasion, we recorded on tape Victor and Pablo´s work (the third student in that group abandoned the course). Because Victor was playing the role of the leader of the group, he tried to explain to Pablo his proposition for a definition of $\lim_{x\to a} f(x) = L$. Victor’s explanation to Pablo is showed in Figure 3.

*"We consider a sequence **a_n**, and its limit **L**, we can put some points here but not necessarily those which we usually draw, but as the last example [given by Mario] that is going down, and some times could be a bit further, but approaching to that number [L], we must begin to say it as 'Adrian' did [about a neighbourhood]. For every **r** or every distance **r** there exists a number around here [pointing to his graph] and here in this case, from this number [n] all the numbers that we obtain comparing to the distance with **L**, that distance is smaller than **r**.*



For all r [distance], exists a value $n \in \mathrm{N}$ *such that* ~~*the distance = a_n*~~ $|L - a_n| < r$

Figure 3: Representations produced by Victor in a peer interaction.

Comparing this definition to the others previously given by Victor, we can appreciate an important cognitive change in his intuitive idea at the beginning of the course and what he expressed in Figure 7. A careful analysis of Victor's oral and written explanations to Pablo suggests that he had a coherent definition in mind, i.e. Victor expressed his conception of convergence using different representations (see Figure 4).

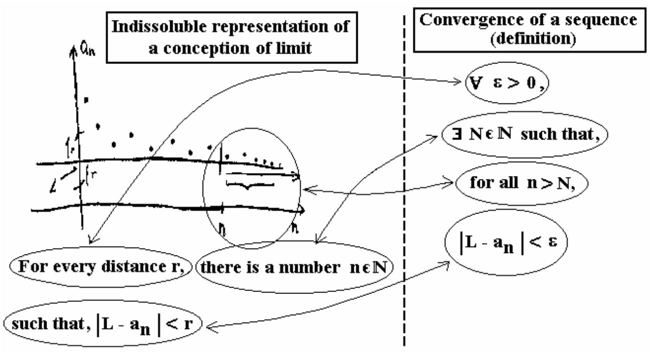


Figure 4: Interpretation of Victor's representations.

With this, we would like to highlight four aspects:

– Victor's ideas emerged in a scientific debate (socio constructivist environment) and in a discussion in his small group;

– Victor's conception evolved and this can be observed throughout his functional representations that played an important role when communicating his ideas to Pablo and in their construction of the concept of limit;

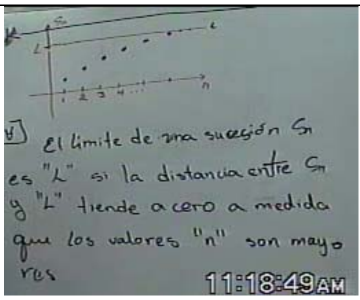
– Victor's functional representations are different from what we find in books or similar sources;

– Victor has constructed a coherent blend of functional representations.

Mise en forme : Puces et numéros

5. How stable is Victor's construction of the concept of limit?

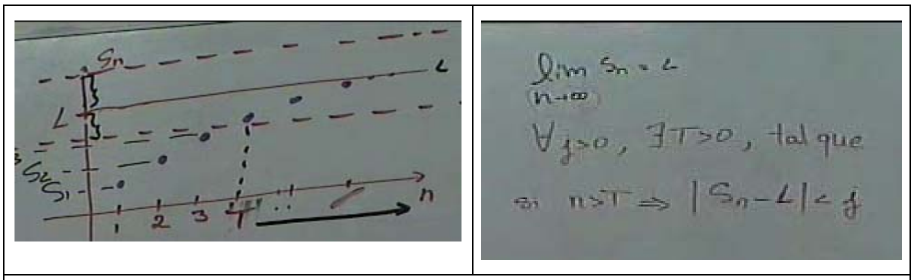
We continued with the teaching experiment and at the end of the course we interviewed all the students. Because we wanted to know whether the students' construction of the concept of limit was stable, we asked them, anew, for a definition. What is interesting here is that Victor begun again with an intuitive idea of limit expressing it in terms of natural language and a graph (see Figure 5).



"The limit of a sequence s_n is "L" if the distance between s_n and "L" tend to zero when the values for 'n' are big."

Figure 5: Victor's verbal definition one month an a half later.

Immediately, he passed to the formal definition beginning with a graphic representation before giving the formal definition (see Figure 6).



$\lim_{n\to\infty} S_n = L \qquad \forall\, j > 0, \exists\, T > 0,$ ***such that if*** $n > T \;\Rightarrow\; |S_n - L| < j$

Figure 6: Victor's formal definition.

Victor used another notation "(j, T)" instead of the more standard notation "(ε, δ)".

We could have stopped here like many researchers when interviewing students, but we thought that to show stability of knowledge, when interviewing students, it was important to present the problem from different perspectives. Therefore, to verify the stability of Victor's cognitive construction, we asked Victor for the negation of the definition and also to give us an example, using the negation, of a divergent sequence.

The reason why we asked this was that we believe that in the construction of mathematical concepts negation and counterexamples play a major role in the construction of mathematical concepts.

Let us elaborate a little bit on this issue. Since the publication of Lakatos (1976) on *"Proofs and refutations"*, a lot of work has been done in connection with this important issue (see for example la preuve: http://www.lettredelapreuve.it/). Some authors like Selden and Selden (1998) consider that "Since success in mathematics, especially at the advanced undergraduate and graduate levels appears to be associated with the ability to generate examples and counterexamples, what is the best way to develop this ability?" They think that it is important to ask students to generate examples; however, we also believe, that it is important to produce conjectures and if necessary, to generate counter examples. Indeed, if we analyze the beginnings of mathematics as a deductive science based on axioms and theorems, the principle of the third exclude and proofs by contradiction are in the genesis of mathematics. If we analyze the history of mathematical ideas, we can verify immediately that the productions of conjectures and counterexamples are key components of the mathematical activity. Based on the above, we asked Victor to write down the negation of his definition and to give us un example of a divergent sequence.

Victor wrote the following:

$$\exists\, j > 0 \text{ such that } \forall\, T > 0, \text{ if } \; n > T \; \text{ then } \; \left|S_n - L\right| > j$$

What is interesting here is that he was not sure about his statement and he begun to construct an example of a divergent sequence and to analyze his statement from that point of view. Then, he constructed the following divergent sequence: $s_1=1$, $s_2=0$, $s_3=1$, $s_4=0$,… (see Figure 7).

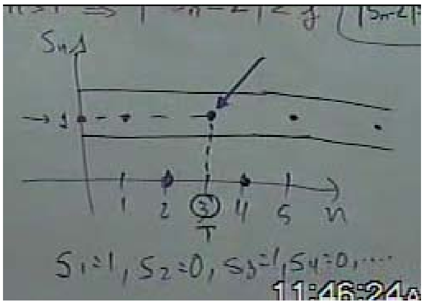


Figure 7: Victor's example of a divergent sequence.

After giving this example, he said that something was wrong with the negation of the definition: "I cannot find where the problem is". The interviewer asked him to pay attention to the quantifiers in the negation he provided and to verify his proposition following the example. It was at that moment that Victor realised that in his definition of divergence a quantifier was missing, he then wrote:

$$\exists\, j > 0 \text{ such that } \forall\, T > 0, \; \exists\; n > T \; \text{ then } \; \left|S_n - L\right| > j\,.$$

6. Discussion

We wanted to show that our approach focused on the functionality of representations to understand the conceptions of the students in the process of constructing concepts. We presented two examples, one to show how institutional representations influence the building of certain conceptions; the other one was chosen to show how a conception evolves throughout a teaching experiment. In Hitt (2003) we presented several examples about the conceptions that some high school teachers have, related to the same teaching experiment. We showed among others a discussion among the same group of teachers when one of them was proposing to change the notation of convergence $\underset{n \to \infty}{Lim}\, a_n \to L$ instead of the institutional one $\lim_{n \to \infty} a_n = L$. The fuctional representation the student (high school teacher) exhibit comes from the intuitive idea of infinity (potential infinity) and the discussion of that idea came up several times during the debate discussions because there were not a consensus to use that notation. From this study and in order to understand their conceptions, we think we need to look closely to those functional representations students come up with when solving a mathematical task.

Our method has worked out with almost all students like with Victor in a socio-construction of a concept. But it was not the case with others like Pedro. In the interview Pedro showed inconsistencies about analysing convergent and also divergent ones.

Our reflection is that Victor´s background in mathematics and informatics might have been a crucial factor that influenced his approach to construct concepts by taking into account several representations and constructing an articulation among them. Pedro's background was in mathematics, but he showed a tendency to memorize definitions and proofs. When asked to work in-group, usually Pedro always tried to solve the task without discussing with his peer students. We think that the methodology doesn't work in Pedro's case. It seems he likes to work alone and it was difficult for him to communicate with his peer students. His approach was always formal and it seems that he can often achieve good performance when dealing with routines problems.

We think we need to teach in a way that permits to extend students' functional representations and try not to impose the institutional ones before time. We propose a model (see Hitt & Paez, 2004; Hitt & Borbon, 2004) where it is important to take into account functional representations in the construction of a concept using a methodology we named ACODESA (collaborative learning and scientific debate environment and self reflection). With this methodology, throughout an evolution of students' conceptions, we are trying to promote conciliation among functional representations students have with the institutional representations we use when teaching.

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