Mathematics as the Science of Patterns - A Guideline for Developing Mathematics Education from Early Childhood to Adulthood

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In memoriam Hans Freudenthal

The aim of the present paper is to give an account of the holistic approach to mathematical education developed in the project “mathe 2000”. The emphasis is on the mathematical roots. It will be shown how a special view of mathematics as the science of patterns can be made practical in developmental research. The paper consists of three sections: A short introduction into the project’s philosophy in the first section is followed by the description of some typical learning environments. In the third section some underlying theoretical principles are explained by referring to the learning environments of section 2.

1. The project „mathe 2000“

In 1985 the State of Nordrhein-Westfalen adopted a new syllabus for mathematics at the primary level (grades 1 to 4). This syllabus coined by Heinrich Winter, the German Freudenthal, marked an important turning point in the history of mathematical education in Germany and exerted an enormous influence on the mathematical education of all levels. Three innovations are particularly remarkable:

(1) A prominent role is given to the four so-called “general” objectives of mathematics teaching which reflect basic components of doing mathematics *at all levels:* ”mathematizing”, ”exploring”, ”reasoning” and ”communicating” (Winter 1975).

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(2) In a special paragraph the complementarity of the pure and the applied aspect of mathematics is stated explicitly, and its consequences for teaching are described in detail.

(3) The principle of learning by discovery is explicitly pre-scribed as *the* basic principle of teaching and learning.

In order to support teachers in putting this syllabus into practice ”mathe 2000” was founded in 1987 as a developmental research project with a clear practical orientation: according to the conception of mathematics education as a ”design science” (Wittmann 1995) the following core areas have been closely linked and pursued simultaneously: the design and the implementation of coherent sets of substantial learning (learning trajectories) as the major parts, pre-service and in-service teacher education in both mathematics and didactics, empirical studies into children‘s thinking and into the communication in the classroom, as well as counseling.[2]

The main source of the developmental research in “mathe 2000” is mathematics - in clear distance from other lines of research in mathematics education including the international movement of “measurable standards”[3] which are based on psychology, cognitive science and general education. “mathe 2000” has adopted an understanding of mathematics as the science of patterns (Sawyer 1955, Devlin 1995), however, with an important additional accent: what matters is not the science of ready-made and static patterns but the vital science of dynamic patterns which can be developed globally in the curriculum as well as explored, continued, re-shaped, and invented locally by the learners themselves. In other words: long-term and short-term mathematical processes count much more than the finished products. The work of British, Scottish, Dutch and Japanese mathematics educators in the sixties and seventies as well as the pace-setting work by Heinrich Winter has served as a model (Fletcher 1965, Wheeler 1967, IOWO 1976, Becker Shimada 1997, Winter 1984, 1987, 1989).

Like the developmental research undertaken in other projects, in particular at the IOWO in Utrecht under Hans Freudenthal’s guidance (cf., IOWO 1976) and recently at the

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[2] For an overview of „mathe 2000“ see http://www.uni-dortmund.de/mathe2000/

[3] For a critical review of PISA see Müller, G.N., Steinbring, H. and Wittmann, E.Ch.: Jenseits von PISA. Bildungsreform als Unterrichtsreform. [Beyond PISA. Educational Reform as Reform of Teaching and Learning]. Seelze: Kallmeyer 2002, Paperback 96 p.

CREM in Nivelles (cf., Ballieu & Guissard 2004), "mathe 2000" has been aiming at a coherent and consistent program for the whole spectrum K – 12 and teacher education. So far the curriculum development has focused on the Kindergarten and primary level as well as on teacher education for the primary level. For this reason the majority of learning environments in the following section is taken from these areas.

2. Typical examples of substantial learning environments

In Wittmann (2002, 2) a substantial learning environment has been defined as follows:

(1) It represents central objectives, contents and principles of teaching mathematics at a certain level.

(2) It is related to significant mathematical contents, procedures and processes ***beyond*** *this level, and is a rich source of mathematical activities.*

(3) It is flexible and can be adapted to the special conditions of a classroom.

(4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.

The first two properties ensure that an SLE is firmly rooted in both the curriculum and in elementary mathematics from an "advanced point of view", the last one guarantees that it reflects the processes of teaching and learning a comprehensive way. The adjective "substantial" refers to *mathematical* substance.

When constructing an SLE the designer is guided by the natural flow of mathematical activity: the starting point of an investigation is always a real or mathematical situation. In the first phase this situation is mathematized or embedded in a broader mathematical framework. Then the emerging structures are explored experimentally with the aim to find patterns and solutions. If tentative patterns have been confirmed by various checks reasoning is called for in order to explain and validate the patterns and the solutions. The last phase of the mathematical process consists of communicating the results, orally or in written form. Obviously Winter's four general objectives reflect exactly these four phases perfectly. That is the reason why they are so important.

In "mathe 2000" the design of SLEs is at the same time consciously guided by the definite intention to integrate the practice of basic skills into the investigation of patterns. As will be shown in section 3 this aspect is crucial for a successful implementation of any innovative program.

How this design philosophy is brought to bear will be illustrated in the next section by means of some typical trajectories of substantial learning environments which are taken mainly from two "mathe 2000" materials: "Das kleine Zahlenbuch" written for early mathematical education in Kindergarten (Müller&Wittmann 2002/2004) and "Das Zahlenbuch", a textbook for grades 1 to 4 (Wittmann&Müller 2004/2005).

2.1 The race to 20 and some variations

Vol. 1 of "Das kleine Zahlenbuch" contains a simple version of the well-known race to 20: A line of circles is numbered from 1 to 10 (another one from 1 to 12). The first player starts by putting 1 or 2 counters on the first circle or the first two circles, the second player follows by putting 1 or 2 counters on the next circles similarly. Continuing in this way the players take turns until one of them arrives at the target and in doing so wins the game.

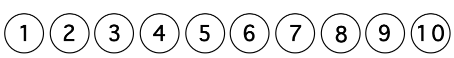

Fig. 1

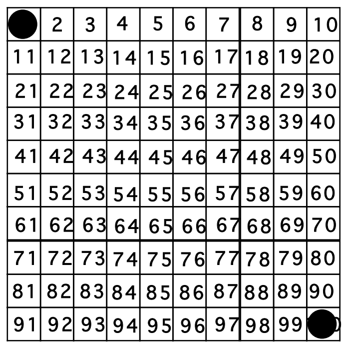

Fig. 2

The „race to 10" helps to found basic arithmetical ideas, for example relationships of numbers on the number line, addition, and repeated addition. While playing the game repeatedly children get more and more familiar not only with the number line but also with the mathematical structure of the game. By analyzing the moves backwards children realize step by step and more or less that the positions 7, 4 and 1 are winning positions. So the first player has a winning strategy: In the first move of the race to 10 she puts down two counters and then responds to a 2-counters move of the second player with a 1-counter move and to a 1-counter move of her opponent with a 2-counters move. In this way the first player jumps from one winning position to the next one and finally arrives at 10. In the race to 12 the winning positions are 9, 6, 3.

The race to 10 is a fully-fledged piece of mathematics. In fact the basic ideas of analyzing these games can be generalized to the wider class of finite deterministic games of strategy for two persons with full information which cannot end in a draw: by means of the game tree and the marking algorithm one can prove that for each of these games there exists a winning strategy either for the first or the second player.

As our empirical studies show 4 to 5year-old children play this game with great pleasure and develop some first insight into the winning strategy. Generally, this knowledge is fairly instable at this level. A few days later most children have to re-discover what they seemed to have mastered before.

In grade 1 the game is re-visited with targets up to 20. In grade 2 a variation is played on the hundred chart (Fig. 2): At the beginning one player puts a red counter on the number 1, her partner puts a blue counter on the number 100. Now the players take turn: the red counter is moved up 1, 2, 10 or 20 fields, the blue counter is moved down 1, 2, 10 or 20 fields. As an additional rule it is stated that the red counter must always cover a number which is smaller than the number covered by the blue counter. Therefore the game stops when the two counters meet. The last player who is able to move is the winner.

The gap between 1 and 100 consists of 98 numbers and as 98 is congruent 2 mod 3 the first player has a winning strategy: the red counter is first moved to 3 or the blue counter to 98. Then the gap consists of 96 numbers and 96 is divisible by 3. As 1, 2, 10 and 20 are not divisible by 3, and the sums 1 + 2 = 2 + 1 = 3, 10 + 2 = 2 + 10 = 12, 10 + 20 = 20 + 10 = 30 and 20 + 1 = 1 + 20 = 21 are all divisible by 3 the first player can always manage to leave a gap which is divisible by 3 while the second player can't. Each move reduces the gap which finally must become 0. As 0 is a multiple of 3 this position is reached by the first player – if she sticks to the winning strategy.

In grade 3 a generalized version of the game is played on the "thousand book": The red counter starts from 1, the blue one from 1000. The allowed moves are +1, +2, +10, +20,

+ 100 or + 200 resp. –1, –2, –10, –20, – 100 or – 200. The winning strategy is analogous to the winning strategy for the hundred chart.

In grade 5 all games will be revisited and analyzed by means of the divisibility arguments mentioned above.

2. 2 Patterns of counters

Vol 1. of "Das kleine Zahlenbuch" contains also the following game: Children are presented series of red and blue counters which follow certain rules (Fig. 3).

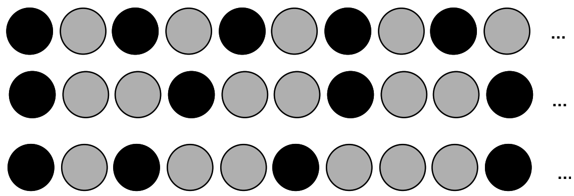

Fig. 3

This game was inspired by the following section from Richard Feyman's address when he was awarded the Nobel Prize for Physics in 1965 (Feynman 1969):

> *When I was very young – the earliest story I know – when I still ate in a high chair, my father would play a game with me after dinner. He had brought a whole lot of old rectangular bathroom floor tiles from some place in Long Island City. We set them up on end, one next to the other, and I was allowed to push the end one and watch the whole thing to go down. So far so good. Next, the game improved. The tiles were different colors. I must put one white, two blues, one white, two blues, and another white and then tow blues – I may want to put another blue, but it must be a white. You recognize already the usual insidiousness: first delight him with play, and then slowly inject material of educational value. My mother who is a much more feeling woman began to realize the insidiousness of his efforts and said: "Mel please, let the poor child put a blue tile if he wants to." My father said; "No, I want him to pay attention to patterns. It is the only thing I can do that is mathematics at this earliest level.*

Our empirical studies show that most children need time to understand what it means to follow rules and to stick to them. If they have reached this level they like to invent their own patterns, however, many of them tend to change their rule while forming a sequence – in particular if the game is played with a partner whose job is to discover the rule.

The construction of sequences according to given rules is a basic mathematical idea which permeates all mathematics. Therefore sequences occur again and again in all curricula.

2.3 Odd and even numbers

Counters are a fundamental means of representing numbers. Usually they are seen as teaching aids. However, their status is primarily not a didactic, but an epistemological one: Greek arithmetic at the times of Pythagoras underwent a period which is called "ψηφοι-arithmetic" and can be considered as the cradle of arithmetic (Becker 1954, 34-41, Damerow/Lefèbre 1981). "ψηφοι" is the Greek word for little stones which the ancient Greek mathematicians used for representing numbers and classes of numbers. For example, they represented even numbers by double rows of stones, and odd numbers by double rows and a singleton. More complex patterns define other classes of numbers, the so-called "figurate numbers": triangular numbers, square numbers, pentagonal numbers, etc.

In the "mathe 2000" curriculum odd and even numbers are introduced in grade 1 in the old Greek way by means of counters (Fig. 4).

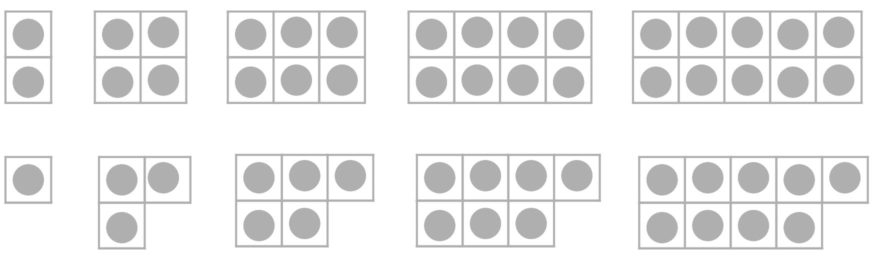

Fig. 4

These patterns are printed on cardboard so that children can combine them and form sums.

The first exercises are to make the children familiar with the material. Based on these experiences children are then asked to find sums with an *even* result. This task is a first suggestion to look at the structure more carefully. The subsequent task is more direct: After calculating the four packages of sums in Fig. 5 children are asked “What do you notice? Can you explain it?”

4 + 6 =	5 + 1 =	2 + 1 =	1 + 8 =
6 + 8 =	7 + 3 =	4 + 3 =	3 + 6 =
8 + 4 =	9 + 5 =	6 + 5 =	5 + 4 =
10 + 2 =	5 + 7 =	8 + 7 =	7 + 2 =
12 + 8 =	9 + 9 =	10 + 9 =	9 + 0 =

Fig. 5

At this early level teachers are expected to refrain from pushing the children. All that teachers have to do is to listen to children’s spontaneous attempts of making some sense of the data.

In grades 2 and 3 even and odd numbers are revisited in wider number spaces. Children are given packages of exercises similar to those in Fig. 5 with bigger numbers and asked the same questions. At this level the even/odd patterns are recognized and expressed in the children’s own words. The teacher’s manual strictly recommends to be content with children’s spontaneous explanations and not try to enforce a proof.

In grade 4, however, children have enough experience with even and odd numbers so that the following task can be set which explicitly demands a proof:

Even numbers can be represented by double rows, odd numbers by double rows and a singleton.

Use this representation to prove that

a) the sum of two even numbers is always even,

b) the sum of two odd numbers is always even,

c) the sum of an even and an odd number is always odd.

Children realize that no singletons occur when even patterns are combined, that in the case of two odd patterns the two singletons form a pair and yield again an even result. Furthermore children recognize that the singleton is preserved if an even and an odd pattern are combined and that in this case the result is odd. The teacher’s task is to take up children’s attempts and to assist them in formulating coherent lines of argument.

As the usual formal proof expresses exactly these relationships in the language of algebra it is well prepared by operations with patterns of counters.

2.4 Arithmogons

This very substantial learning environment was stimulated by a wonderful article which was published 30 years ago (McIntosh&Quadling 1975). The authors' original geometric setting has been somewhat changed in order to allow for using counters (Fig. 6): A triangle is divided in three fields. We put counters or write numbers in these fields. The simple rule is as follows: Add the numbers in two adjacent fields and write the sum in the box of the corresponding side.

Various problems arise from this context: When starting from the numbers inside, the numbers outside can be obtained by addition. When one or two numbers inside and respectively two numbers or one number outside are given the missing numbers can be calculated by addition or subtraction.

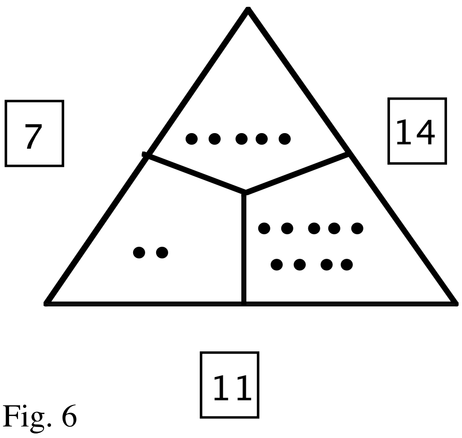

Fig. 6

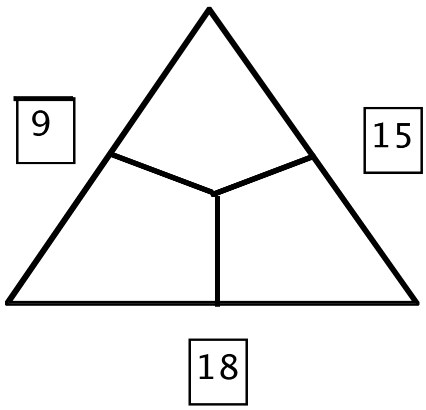

Fig. 7

When the three numbers outside are given (Fig. 7), an immediate calculation is not possible, so some thinking is required. Firstgraders can find the solution by (more or less) systematically varying the number of counters in the inner fields. There are, however, also systematic solutions which arise in the course of a continued study of arithmogons in the following grades.

In grade 4 children are guided to discover a systematic solution: In the first step they complete some arithmogons and calculate the sum of the numbers inside and the sum of the numbers outside. They discover that the latter sum is double as much as the first one and prove it by pointing out that each inside number contributes to two outside numbers. In the second step students are asked to subtract an outside number from the

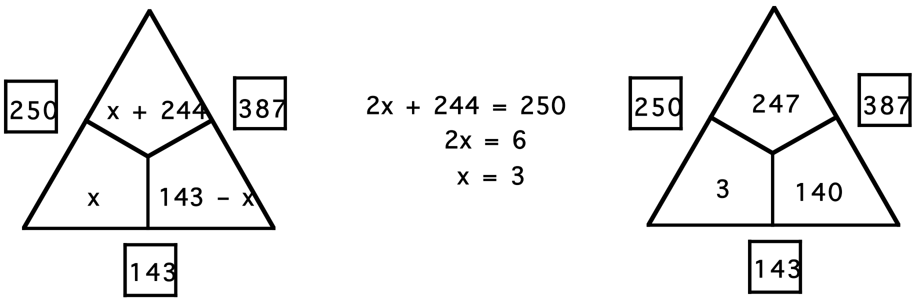

Fig. 8

sum of the inside number and discover that the result is the inside number opposite to the outside number. The two first steps can be summarized to determine the inside numbers from the outside numbers.

At the secondary level arithmogons will be used to solve arithmogons by means of a linear equation. The early method of systematically varying the numbers is a very good help for finding the equation (Fig. 8).

Arithmogons can be extended to quadrilaterals, where new phenomena arise: Either we have more than one solution (Fig. 9) or no solution (Fig. 10). For the existence of solutions it is necessary and sufficient that the sums of opposite numbers are equal. (Each of these sums is the sum of all inner numbers.)

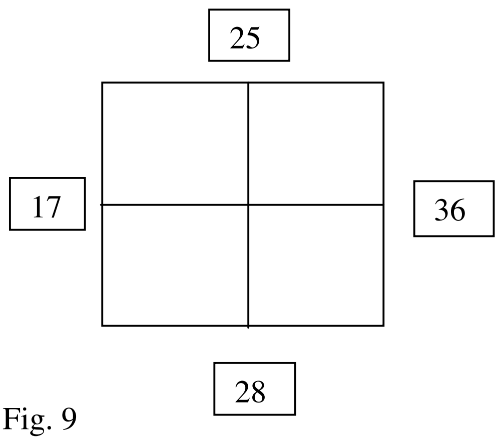

Fig. 9

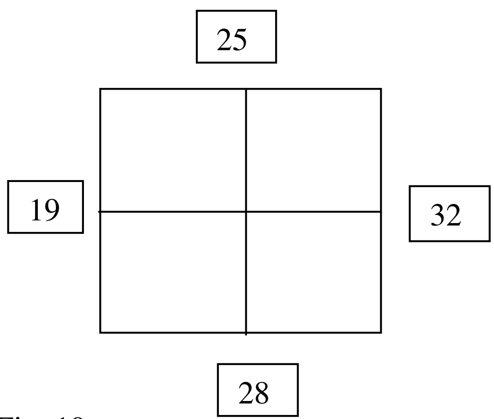

Fig. 10

Of course arithmogons can also be generalised to polygons with n sides.
The mathematics behind arithmogons is quite advanced: the inner and outer numbers can be written as vectors, and the relationship between them is a linear mapping from R^n into R^n. Interestingly the corresponding matrix is non-singular for odd n and has the rank (n–1) for even n (McIntosh&Quadling 1975).

The two-dimensional structure of arithmogons can be generalized to "arithmohedra" in an obvious way: Numbers are assigned to vertices and faces of a polyhedron such that the numbers assigned to the faces are the sum of the numbers assigned to the vertices of this face. In this way a rich variety of examples can be created. This material can be explored by College students and student teachers in a course on Linear Algebra with great gains. All phenomena and all concepts relevant for the theory of systems of linear equations occur and can be explained in this context, up to Steinitz' theorem and the dimension theorem. For student teachers it is most important to see their mathematical education in the professional context as will be explained in section 3.

2.5 ANNA Numbers

Four-digit palindromes like 6446, 1221, 7007, are called ANNA numbers. For any ANNA number there is a natural partner with the same digits, for example 2332 and 3223, 5885 and 8558. A nice piece of mathematics arises from the following simple exercise: Fourthgraders are asked to choose two digits, to form the two possible ANNA numbers and to subtract the smaller number from the bigger one. When the results of the calculations obtained by the children are collected, checked, improved, and ordered it turns out that only a few results are possible: 891, 1782, 2673, 3564, 4455, 5346, 6237, 7128, 8019 (and possibly 0 if numbers like 2222 are accepted as ANNA numbers). The sequence of these results contains not only interesting patterns as far as the sequences of the place values are concerned: surprisingly, all numbers are multiples of 891.
The structure becomes even richer when the tasks are assigned to the results. For example, the result 2673 belongs to tasks like 5225 – 2552, 7447 – 4774, 4114 – 1441 etc. that is to all tasks for which the difference of the digits is 3. It turns out that in

general the difference of two ANNA numbers with the same digits is 891 times the difference of their digits.

This phenomenon can be proved and explained in various ways by using representations with which students are familiar. One proof uses the place value table as follows:

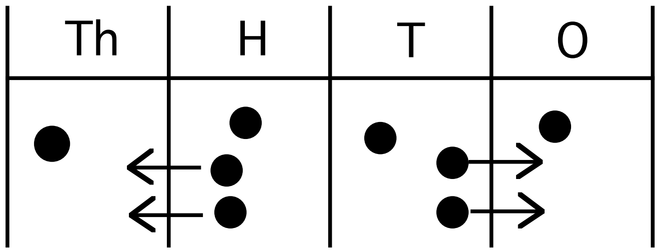

Fig. 11

In order to get from 3443 to 4334 on the place value table one counter from the hundreds column has to be pushed to the thousands column and one counter from the tens column has to be pushed to the ones column. So the difference of the numbers is 1000 – 100 – 10 + 1 = 891. These operations can be applied to all ANNA numbers whose digits differ by 1. If the difference d between the digits is bigger than 1, then d pairs of counters have to be moved from the hundreds and tens columns to the thousands and ones columns. This means that the difference between these ANNA numbers is d times 891 (see Fig. 11 for d = 2).

A similar investigation can be conducted for NANA numbers. Here all differences are multiples of 1000 – 100 + 10 – 1 = 909.

ANNA and NANA numbers do not come out of the blue in grade 4. They are well prepared by similar activities with ordinary two-digit numbers in grade 2 and three-digit palindromes in grade 3 (IMI numbers). Here the possible results are multiples of 9 and 91.

In grade 5 the place value table will be used in this way to prove the rule of the divisibility by 9 and 3 as shown by Winter 1985.

2.6 Fitting polygons

A fundamental idea of elementary geometry is „Fitting". Freudenthal (1969, 422-23) describes it as follows:

In paving a floor with congruent tiles there is a leading idea, I mean fitting. It is the same as in space and it is realized as concretely. Fitting is a motor sensation. Psychologists can tell you how strongly the motor component of the personality is marked at a young age, how important motor apprehension and memory may be. Things fit. Do children ask why? Apart from a rare exception young children do not. All these miracles of our space do not seem to make any impression. But they grind as millstones. The highest pedagogical virtue is patience. One day the child will ask why, and there is no use starting systematic geometry before that day has come. Even worse: it can really do harm.

Children decompose square paper in two or four parts by folding and cutting along axes of symmetry and re-arrange the parts in various ways. Many of the new figures will become important later in the curriculum. For example: The four isosceles rectangular triangles of two small squares can be put together to make one big square - a special case of the Pythagorean theorem.
To develop „fitting“ over the grades the “mathe 2000” curriculum starts in grade 1 with paper and scissors activities (Fig. 12). A square is decomposed into two or four isosceles right triangles and the parts are recombined to make other figures.

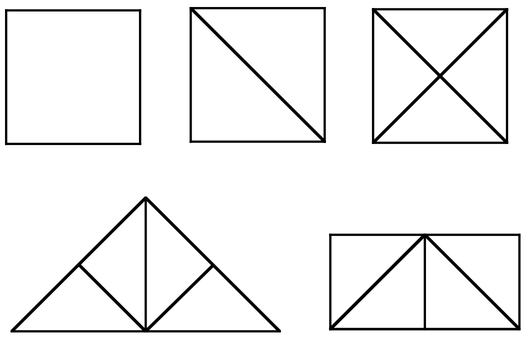

Fig. 12

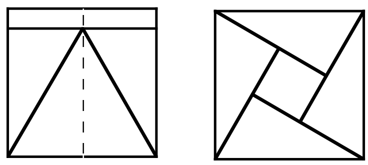

Fig. 13

In grade 2 this activity is extended: square paper is folded and cut such that equilateral triangles and halves of equilateral triangles are obtained (Fig. 13). One of the figures which can be made by these forms is well-known as a foundation of the Pythagorean theorem.

Fitting regular polygons is done in grade 3 by means of a template which allows for drawing squares, regular triangles, pentagons, hexagons and octagons with the same side length. Children can explore experimentally which figures fit which way. They realize that there are only three regular tessellations and discover some semi-regular tessellations.

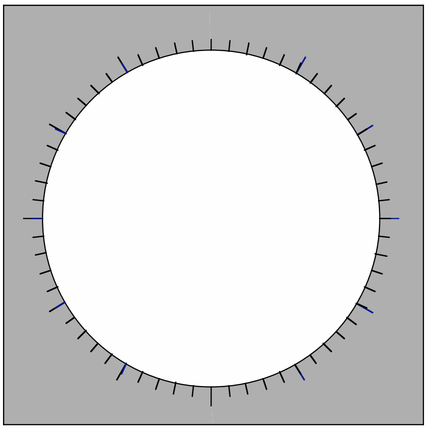

Fig. 14

In grade 4 children make regular polygons from cardboard by means of the „drawing clock" (Fig. 14) and build the five Platonic solids. The name „drawing clock" is derived from the fact that a circle is divided into 60 equal parts. As 60 is divisible by 3, 4, 5 and 6 the drawing clock allows for a convenient construction of squares, regular triangles, pentagons, hexagons and octagons. For example, to draw a regular polygon one has to divide the circumference in five equal parts of "12 minutes" each and connect the points. When drawing clocks of different sizes are used polygons of different sizes are obtained. The shapes are copied on cardboard. The circular segments attached to the sides of the polygons can be folded down and used as to paste the polygons together. In this way children can make stable models of all five Platonic solids. The proof of the existence of at most five Platonic solids at the end of book 13 of Euclid's "Elements of Mathematics" is fully in line with children's arguments.
In grade 5 cutting and fitting experiments with polygons will be used to found the concept and the measure of an angle what again corresponds to the historic development (Becker 1954, 27). In the following grades cutting polygons into pieces and re-combining these pieces is the usual way to the area formulas and to the Pythagorean

theorem (Wittmann 1995, 134 – 136). In the “mathe 2000” curriculum it is well prepared by specific activities which start in grade 1.

2.7 Conic sections

In an address at the beginning of the 20th century J.J. Sylvester stated:

> *The discovery of the conic sections, attributed to Plato, first threw open higher species of form to the contemplation of geometers. But for this discovery, which was probably regarded in Plato's time and long after him, as the unprofitable amusement of a speculative brain, the whole course of practical philosophy of the present day, of the science of astronomy, of the theory of projectiles, of the art of navigation, might have run in a different channel; and the greatest discovery that has ever been made in the history of the world, the law of universal gravitation, with its innumerable direct and indirect consequences and applications to every department of human research and industry, might never to this hour have been elicited.*

It is one of the clear signs of the decline of mathematical education in the past decades that conics have been more or less eliminated from the curricula. In the “mathe 2000” curriculum an attempt will be made to re-install them as far as the boundary conditions permit.

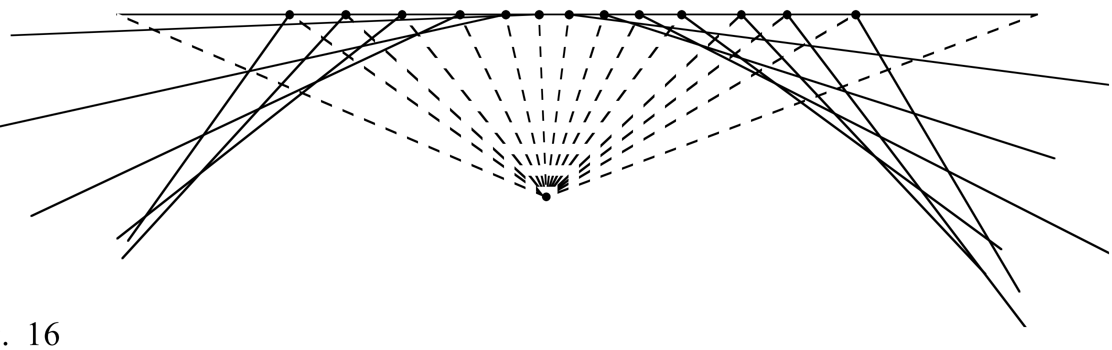

Fig. 16

The treatment of conics can start already in the primary school. In grade 4 of the “mathe 2000” curriculum the pedal construction of a parabola is introduced as an exercise in the use of a ruler (Fig. 16). Students are stimulated to observe what happens if the position of the focus is changed. Some children get excited with this construction.

In grade 5 the envelope constructions of the conics will be introduced similarly via exercises for constructing the midpoint and the perpendicular bisector of a segment. A parabola is defined as the locus of all points which have the same distance from a point F and a line d. The construction arising from this definition is very simple (Fig. 17):

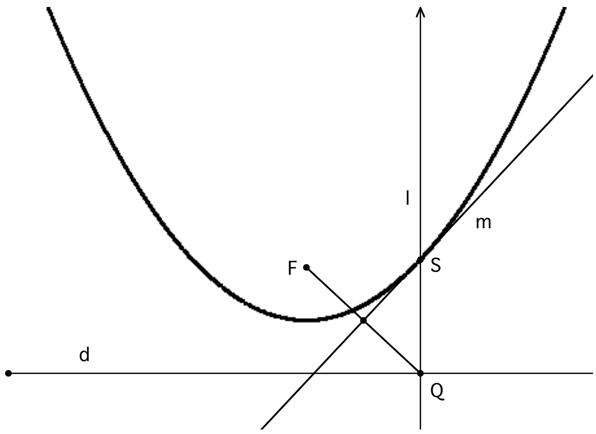

Fig. 17

First a line d is drawn and a point F outside this line is selected. Then the following construction is repeated for a set of points Q on d: For each segment FQ the perpendicular bisector m is drawn and intersected with the perpendicular line l erected on d in Q. By construction the point of intersection S has equal distance from F and d.
The existing software for dynamic geometry is an ideal tool in order to draw and animate the locus of S.
Based on the observed phenomena the focal property of a parabola can be explained, that is proved, in later years: The point S is the only point on m and on the parabola, because the distance of all other points on m from d is smaller than their distance from F. In analogy to the tangents of a circle the line m is called a tangent to the parabola. Now, a ray of light which is emitted from F and hits the parabola in S is reflected as if it would hit the plane mirror m. Therefore the reflected ray seems to come from the mirror image Q of F, that is, the reflected ray coincides with l. So all reflected rays leave the parabolic mirror as a bundle of parallel rays.
In a similar way the other conics can be defined and explored. The case of the ellipse is particularly interesting. It plays a crucial role in Feynman's lost and rediscovered lecture in which he derived the Kepler ellipses by revising the derivation in Newton's "Principia mathematica" (Goodstein/Goodstein 1996). Feynman's approach is elementary and can be integrated into the upper secondary curriculum as well as into

teacher education. In this way students and student teachers can become familiar with Newton's epoch-making achievement.

3. Theoretical Reflections

The old Greek "θεορια"means "holistic view" and in this sense the role of theory consists of establishing common links within a field of experience. Accordingly the following theoretical reflections refer to principles underlying the design of the learning environments of the preceding section.

3.1 The quasi-empirical nature of mathematics and the selection of representations

The most important common feature of these learning environments is the following one: the activities in exploring of patterns and finding proofs depend on appropriate representations of mathematical objects in question. It was I. Lakatos who in his masterpiece "Proofs and Refutations" (Lakatos 1979) first pointed to the fact that mathematical theories are always developed in close relationship with the construction of the objects to which they refer. Graph theory grows with the construction of graphs, group theory grows with the construction of groups, theories of coding with the construction of new codes, and so on. These mathematical objects form a kind of "quasi-reality" which permits the researcher to conduct experiments similar to experiments in science (see also Dörfler 1991). The availability of computers has greatly enhanced the access to these "quasi-realities".

In a recent paper an eminent research mathematician wrote (Arnold 1998, 229, 233):

> *"Mathematics is a part of physics… I get the impression that mathematicians who have little knowledge of physics believe in the principal difference of axiomatic mathematics from the modeling which is common in natural science and which always requires the subsequent checking of deductions by an experiment.*
>
> *Every working mathematician knows that, without some form of control (best of all by examples), after some ten pages half the signs in formulae will be wrong and twos will find their way from denominators into numerators. The technology of combating such errors is the same external control by experiments or*

observations as is to be found in any experimental science and it should be taught to all juniors in school."

In general education *informal* representations of mathematical objects are indispensable as they provide a "quasi-reality" which is easier to grasp and to work on than symbolic representations. An important role is played by geometric representations as geometry is a language which mediates between natural language and algebra (Thom 1973, 206 - 209). The selection of representations which incorporate fundamental mathematical structures needs careful consideration (see Wittmann 1998 for arithmetic). The learning environments in section 3 are based on the number line, counters, calculations with numbers, the place value table, models of polygons, and drawings.

3.2 The double role of representations

Representations of mathematical objects form a kind of interface between pure and applied mathematics. They can be seen as concretizations of abstract mathematical concepts and at the same time as representations of real objects. Compared with the abstract objects these representations are more concrete than the mathematical objects which they represent, and compared with the real objects which they model they are more abstract. Conics provide a striking example for the amphibian-like double role of representations: On the one hand a drawing of an ellipse is a concrete representation of the mathematical object "ellipse" and on the other hand a schematic representation of any real elliptic shape, for example a lithotripter for breaking up kidney stones, a mirror of a telescope or the orbit of a planet.

Another example is provided by counters. On the one hand collections of counters can be considered as *concrete models of abstract numbers*. Operating with counters allows for proving relationships between numbers, for example between even and odd numbers. On the other hand counters can be used *to model real situations*. The following task for first graders is typical:

10 children on a playground have to be assigned to an Indian tent and a climbing stage according to various boundary conditions, for example:

(1) 3 children are playing in the Indian tent.

(2) Half of the children are playing on the climbing stage.

(3) There are two more children playing in the Indian tent than on the climbing stage.

For solving these word problems the Indian tent and the climbing stage are drawn on a sheet of paper, the 10 children are represented by 10 counters and these counters are shuffled around in order to meet the conditions.
Studying mathematical objects via representations which are *possible* models of reality is the best preparation for mathematical applications. Contrary to real objects or models of real situations which are charged with various constraints mathematical objects allow for unlimited operations and for establishing theoretical knowledge which is much more applicable than knowledge directly derived from mathematizing real situations.
From this perspective the widely held view that mathematics which is taught in general education, in particular at the lower levels, should in principle be derived from and closely related to applications is an educational error.

3.3 Operative proofs

By working with appropriate representations of mathematical objects sound proofs of general statements become possible. In the race to a target number (2.1) the winning strategy depends on looking at pairs of moves independently of the special positions. The proof of the theorem about even and odd numbers (2.3) uses operations with double rows and singletons whereby the size of the numbers does not matter. The relationship of the inner and outer numbers of an arithmogon (2.4) does not depend on special numbers but only on the rule for calculating the numbers outside from the numbers inside. The proof of the pattern underlying the possible differences of pairs of ANNA numbers (2.5) is based on operations with the place value table. The tesselations and solids made of regular polygons (2.6) arise from combining polygons and the effects of fitting. At the primary level these effects are taken for granted. Later in the curriculum they will be corroborated by means of the angle concept. The proof (2.7) of the focal properties of the parabola is based on the envelope construction.
The crucial point of this type of proof has been clarified by Jean Piaget in his epistemological analysis of mathematics: Mathematical knowledge is not derived from the objects themselves, but *from operations with objects* in the process of reflective

abstraction („abstraction réfléchissante“, Beth/Piaget 1961, 217-223): When it is intuitively clear that the operations applied to a special object can be transferred to all objects of a certain class to which the special object belongs then the relationships based on these operations are recognized as generally valid. As these proofs draw from the effects of operations on the objects under consideration they are called "operative proofs" (Semadeni 1974, Kirsch 1979, Jahnke 1989, Wittmann 1995, 144- 148, 154-160).

The advantage of operative proofs in the context of education is obvious: These proofs are embedded in the investigation of problems, closely related to the quasi-experimental investigation of patterns, based on the effects of operations, and communicable in a simple problem-oriented language.

ANNA numbers may serve as an example. The conceptual relationship on which the pattern of differences rests can be stated formally in two lines:

If $A > N$ then $(A \cdot 1000 + N \cdot 100 + N \cdot 10 + A) - (N \cdot 1000 + A \cdot 100 + A \cdot 10 + N) = (A - N) \cdot (1000 - 100 - 10 + 1) = (A - N) \cdot 891$.

This proof, however, is useless for primary mathematics and teacher education for this level.

3.4 The epistemological triangle

In his empirical research into teaching/learning processes Heinz Steinbring has introduced the epistemological triangle (Steinbring 2005, 22, Fig. 18). The epistemological triangle is an expression of the fact that learners cannot grasp a mathematical structure solely at the symbolic level. The mathematical concepts carrying

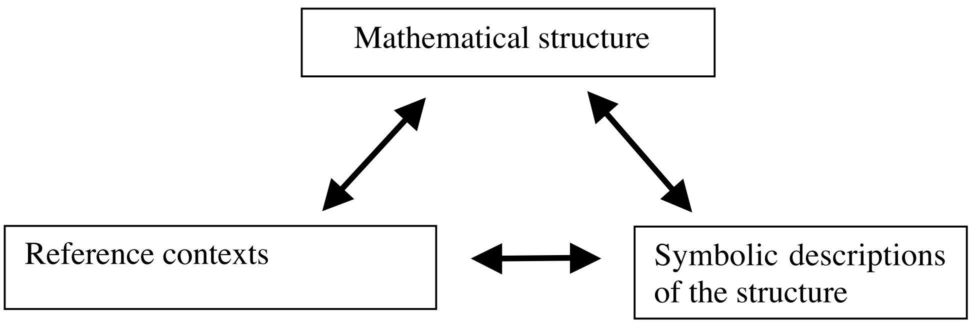

Fig. 18

the structure become meaningful only via reference contexts in which the structure is embodied and which allow for conducting experiments. The experiences gathered in working within reference contexts form the basis for social exchange with the teacher and among the students.

The example of ANNA numbers is good illustration. Children cannot capture the conceptual relationships in their symbolic setting. They need calculations with special ANNA numbers and the place value table as a reference context in order to explore, explain and understand the pattern.

3.5 Productive practice

What really counts for the long-term mastery of a piece of mathematics is not how it is introduced but how it is practiced: *Repetitio est mater studiorum*. The history of mathematics teaching shows that many attempts of reforming mathematics teaching have failed because they neglected the practice of basic skills. Teachers will reject programs in which an essential component of teaching and learning is missing, and they are right to do so. Traditionally the practice of skills has been firmly rooted in the school system and in textbooks in the legendary form of "drill and practice". Developing mathematics as the science of patterns is not compatible with this system. In order to resolve this seeming dilemma a novel approach to practicing skills is needed. In another of his epochal papers Heinrich Winter showed how the practice of skills can be reconciled with the principle of learning by discovery (Winter 1984). In elaborating on Winter's ideas the concept of "productive practice" has been developed and the design of substantial learning environments for practicing skills was put into the very centre of "mathe 2000". For constructing productive exercises one has to look for patterns whose experimental investigation involves the repeated execution of a certain skill. The learning environments "odd and even numbers" (2.3), „arithmogons" (2.4), "ANNA-numbers" (2.5) and "conics" (2.7) are typical examples of „productive practice". The two volumes of the „Handbook of practicing skills" (Wittmann&Müller 1990, 1992) contain coherent sets of substantial learning environments for introducing and practicing the central topics and skills of the primary curriculum like the addition table, multiplication table, informal arithmetic and standard algorithms. It is likely that the success of the "mathe 2000" primary curriculum in several countries is due to this integrative approach.

Besides productive practice the „mathe 2000" curriculum also contains a systematic course in mental arithmetic which is called „Blitzrechnen [Calculightning]".[4] Within a conception of mathematics as the science of patterns a firm mastery of techniques is a sheer necessity for the exploration of patterns,

3.6 Teacher Education

It goes without saying that the reform of mathematics teaching based on a view of mathematics as the science of patterns is greatly supported by a corresponding reform of teacher education as teachers who have acquired first-hand experiences with mathematical processes during their studies are much more likely to carry the reform than teachers who have not.

The question is how teacher education can be organized in order to provide these experiences best. Around the world there is a widely shared view among mathematics educators that mathematics education proper (didactics of mathematics) is the key to reforming teacher education, and therefore the main emphasis is put on courses in didactics. However, there are good reasons to see the mathematical courses as the key of reform. It is well-known that around the world mathematical courses or even whole mathematical programs often make only little or no sense for student teachers: Either the relevant subject matter is not covered at all, or the mathematical substance is stifled by a formal style of presentation or, even worse, there **is** no substance: mathematics is reduced to conceptual or procedural skeletons. This criticism refers also to recent approaches to teacher education which have been explicitly announced as the mathematicians' answer to the problem of providing teachers with the "necessary mathematical knowledge". A typical example is a textbook which has been authorized by the American Mathematical Society (Jensen 2003). According to the style of this book ANNA numbers, for example, would be analyzed, if at all, in the formal way as mentioned at the end of section 3.3.

[4] This course is available on a bilingual CD-ROM (German and English) which was awarded the German Software Prize digita in 1997 (Krauthausen, Müller &Wittmann 1997/1998).

As the “mathe 2000” experience shows teacher education can effectively be improved by linking mathematical courses to substantial learning environments as far as both subject matter and method is concerned. By definition SLEs are based on substantial mathematics *beyond* the school level. Therefore every SLE offers mathematical activities for student teachers on a higher level. However, disconnected mathematical islands attached to scattered SLEs do not serve the purpose properly. What is needed in teacher education are systematic and coherent courses of elementary mathematics which cover the mathematical background of a variety of SLEs. To develop such courses is a challenging problem for the future.

Within “mathe 2000” a special series *Mathematics as a Process* has been started which tries to fill this gap. The first volume “Arithmetik als Prozess” has already appeared (cf., Müller, Steinbring&Wittmann 2004). As the title of the series indicates the emphasis is on mathematical processes. *Informal representations* are preferred to formal ones wherever possible. So student teachers are systematically given the opportunity to learn the mathematical language which they need in their profession, not the language of specialists which is useless or even harmful in communicating with students. To give an example: in “Arithmetik als Prozess” the chapters on elementary number theory up to the “little Fermat” are based exclusively on operations with the number line and with rectangular patterns of dots. All proofs are operative proofs. In the tentative volume “Algebra as a Process” the theory of linear equations will be centred around arithmogons and arithmohedra as indicated in section 2.4.

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