Best Estimate Evaluation in an Economical Framework: Key Points, Best Practices and Pitfalls

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In the last ten years, we have observed a generalization of the use of « economic valuations » in various frameworks used by insurers: regulation (Solvency II), accounting (IFRS) and financial reporting (MCEV).

This led to the use by insurers of methods originally developed for pricing financial instruments to calculate their liabilities. On this occasion, many challenges have emerged, particularly in life insurance:

- long duration of life insurance liabilities;
- no market;
- partially endogenous risk factors;
- volatility of the value which does not reflect the risks carried.

The aim of this presentation is to present the specific elements induced by the use of « economic valuations » in the insurance business and the consequences for valuation of life insurance liabilities.
To implement the standard model of Solvency II, we need to compute the « economic balance sheet »:

- get the prices of different assets at the date of calculation;
- calculate the value (« price ») of liabilities.

To compute the Solvency Capital Requirement (SCR) with the standard formula, we have to recalculate the change in price of assets and liabilities at date 0 when a shock is applied.

For basic assets (equities, sovereign and corporate bonds mainly), prices come directly from the market. For derivative assets (convertible bonds, options, etc.), we should strictly speaking have closed formulas to recalculate prices depending on the level of underlying risk factors (equity, interest rate, credit, liquidity).

Economic Scenario Generator (ESG) provides the answer to the second item.
Cash-flows calculation, then the best estimate that can be deduced, is performed in the following framework:

\[ \Lambda_x = \sum_{t=1}^{\infty} \frac{1}{(1+i)^t} I_{x \leq T} \left( \frac{1}{T_x} \right) \]

\[ \text{BEL} = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{a=1}^{A} \text{Flux}_{t,n,a} - \text{Cotisation}_{t,n,a} + \text{Frais}_{t,n,a} - \text{Chargement}_{t,n,a} \left( 1 + R_n(0,t) \right)^t \]

\[ \rightarrow E_{N \to \infty}^{P \otimes Q F} \left( \sum_{j \geq 0} \frac{F_j}{(1 + R_j)^j} \right) \]

- Economic Scenarios Generator (risk neutral)
- Evaluation of mathematical reserves before revaluation and S1 financial reserves
- Calculation of profit sharing
- Revaluation of liabilities
- Iteration of projection time
- Loops on economical scenarios
Computing the economic balance sheet at time $t=0$ fulfills the requirements of Pillar 1 of Solvency II. To meet the requirements of Pillar 2, one must also be able to project this balance sheet in the future.

An ALM model in life insurance should then be able:

- to compute prices (assets and liabilities);
- to compute quantiles of the distribution of the net asset value, that is prices distributions.

The first item uses risk neutral measure, the second one uses historical measure.
As part of a comprehensive modeling which aims to provide distributions of economic value, we use a two-tiered approach:

- The construction of a functional $g$ providing the vector of prices based on state variables $Y$ to the calculation date, $\pi_0 = g(Y_0)$

- Building a dynamic model for risk factors $Y_t$

We can then determine prices at any time using:

$$\pi_t = g(Y_t)$$

The construction of the functional $g$ is based on the classical assumptions of financial markets including the « no arbitrage assumption », which leads to construct risk neutral probability that make the discounted prices processes martingales.

The construction of the dynamics of $Y$ is a problem of econometrics.
For example, with the classical Vasicek model, the following dynamics are used for the unique risk factor (the short rate):

Quantiles

$$dY_t = dr_t = a(b - r_t)dt + \sigma dW_t$$

Pricing

$$dr_t = a(b_\lambda - r_t)dt + \sigma dW_t^Q$$

and the pricing function is:

$$g \left( r_t \right) = P(r_t, T-t) = \exp \left( \frac{1-e^{-a(T-t)}}{a} \left( r_\infty - r_t \right) - (T-t)r_\infty - \frac{\sigma^2}{4a^3} \left( 1-e^{-a(T-t)} \right)^2 \right)$$

with  

$$r_\infty = b_\lambda - \frac{\sigma^2}{2a^2}$$

We observe here that the link between the two representations is made via the parameter $\lambda$.

Note: the parameter $\sigma$ is theoretically invariant.
In this presentation we will first focus on the « classical » calculation framework used by the (French) industry.

If the principles of those calculations are simple, their effective implementation may be tricky.

Then we will focus on most relevant alternative approaches to the calculation of the second pillar (ORSA).
1. Simulating Best Estimates
2. Computing Best Estimates
3. Application: Balance Sheet Projection
1. Simulating best estimates

Illustration #1: convergence

We consider an unit linked contract with an underlying asset modeled by a log-normal process and the risk free rate modeled by a Vasicek model, so that:

\[ S(t) = S_0 \exp \left( \int_0^t \left( r(u) - \frac{\sigma^2}{2} \right) du + \sigma B(t) \right) \]

\[ dr(t) = k \left( \theta - r(t) \right) dt + \sigma_r dB_r(t) \]

The contract duration is 10 years, fully redeemed at the end. In the meantime, the structural ratchet rate is 2% and a cyclical ratchet rate equals 5% is added when the value of the share at time \( t \) is lower than the initial value.

The best estimate value of the contract is simply the initial value of the share.
1. Simulating best estimates

Illustration #1: convergence

The convergence of the empirical best estimate to its theoretical value is slow and, after 1000 runs, a difference of about 1.5% remains. This difference leads to a difference of about 15% on own funds...

Dividing this difference by 10 multiply the number of simulations by 100.

Therefore it may be useful (essential) to optimize this scheme (see e.g. Nteukam and Planchet [2012]).

1. Simulating best estimates

Illustration #1: convergence

We can also remember that calculating the price of a single ZC by simulation is not simple.

Convergence for this type of asset is slower than for more complex cash-flows such as those of a savings contract.
1. Simulating best estimates

Illustration #2: bias

Euler discretization process of the short rate for, for example, H&W model leads to:

\[
    r_{\delta}(n\delta) = f(0,n\delta) + \frac{\sigma^2}{2k^2}(1 - e^{-kn\delta})^2 + \sigma\sqrt{\delta}e^{-kn\delta}\sum_{i=1}^{n} e^{+k(i-1)\delta}\varepsilon_i
\]

This discretization does not introduce bias for the mean, but:

\[
    V(r_{\delta}(n\delta)) = \frac{\sigma^2}{2k}(1 - e^{-2kn\delta})\times \frac{2\times k \times \delta}{e^{2k\delta} - 1} = V(r(n\delta))\times \frac{2\times k \times \delta}{e^{2k\delta} - 1}
\]

Constant \( c_\delta = \frac{2\times k \times \delta}{e^{2k\delta} - 1} \) tends to 1 when the discretization step tends to 0, but can lead to poor results with a monthly or annual discretization.

It is then necessary to reduce the bias (directly, with a Milstein’s discretization, or an exact one when available, etc.).
1. Simulating best estimates

Illustration #2: bias

For example, we consider the approximation of the price of a ZC bond by simulation. With a sample size of 10,000 we have:

The bias is material, even with a huge sample size.
1. Simulating best estimates

Illustration #2: bias

One can see the same phenomenon with more complex processes, e.g. the calculation of a “real” best estimate.

We must distinguish the cash-flows projection step (usually annual) and the discretization used to approximate the discount factors (see Ifergan [2013]).

In practice in the projection models used to calculate a best estimate, we can consider that the sampling error is of the order of 0.50%.
1. Simulating Best Estimates

2. Computing Best Estimates

3. Application: Balance Sheet Projection
2. Computing best estimates

The previous approaches by projecting the flow of benefits under the contract with Markov models and obtaining numerical results relies heavily on simulation. If it helps describe the flow dynamics accurately, cumbersome calculations make these models difficult to use, configure and maintain. In particular, the use of these approaches within the framework of internal models is particularly difficult (cf. Bauer et al. [2010]).

Markov style models mentioned above are poorly suited to ORSA projections, because of the large computation time needed (it can be optimized, see Nteukam et al. [2014]) and the lack of robustness (which is mainly due to over parameterization).

Thus, our goal is now to build a model able to take into account complex contracts for computing projected best estimates valuations well suited to the ORSA framework.

This model is taken from Bonnin et al. [2014b].
2. Computing best estimates

To achieve this goal, we develop a simple framework to compute a coefficient (with a closed formula) which when applied to the mathematical reserve gives the associated fair value of the contract.

Indeed, in general we observe that the best estimate value is near the mathematical reserve (e.g. between 95% and 105% of it on most cases). Thus we seek a coefficient to be applied to the mathematical reserve that accounts for the time value of options.

For the Solvency Capital Requirement (SCR) calculation and projection, we adapt here the model described in Guibert et al. [2012] to life insurance. The framework is built by directly specifying the dynamics of the increase rate of the contract. In our model the best estimate value of the contract becomes computable and its application in the ORSA framework shows all its interest.

In particular we obtain an explicit expression of the SCR which is easily computable using basic simulation technique.
Consider a savings contract with a surrender value for a policyholder that evolves according to (we denote by $t=0$ the calculation date)

$$VR(t) = VR(0) \times \exp \left( \int_0^t r_s(u) du \right)$$

the value of the mathematical reserve at time $t$ is

$$PM(t) = VR(t) \times \exp \left( -\int_0^t \mu(u) du \right)$$

For the current contract, the payment of the mathematical reserve in case of early withdrawal (ratchet or death) and the term $T$ of the contract, assumed to be fixed (non-random), both determine the benefits of the contract. The flow of updated service contract considered here is simply expressed as a function of $\tau$, the release date (random) of the contract (which is the surrender or death time)

$$\Lambda = VR(\tau \wedge T) \times \delta(\tau \wedge T) \quad \delta(t) = \exp \left( -\int_0^t r(u) du \right)$$
2. Computing best estimates

The main idea is to consider that the accumulation rate is affected by two kinds of randomness:

- An hedgeable hazard linked with the market price of the assets;

- Corrections to this return by piloting the accounting result. On this point, even if the management actions are deterministic, we can consider that there is a source of randomness (not hedgeable) associated with the moment the unrealized profit and loss are booked. Indeed, the book yield of a transfer of assets depends on the market price of the asset but also its cost. This second source of randomness must be introduced into the model.

The proposed model is also the following:

\[ r_s(t) = r(t) + \omega(t) \]

with the short interest rate \( r \) the hedgeable part of risk and \( \omega \) the non-hedgeable one.
2. Computing best estimates

By definition, the best estimate at time $t=0$ of the contract is calculated via

$$BEL(0, T) = E^{P^{nh} \otimes Q^{h}}(\Lambda)$$

with the historical probability $P^{nh}$ modeling the non-headgeable risks and $Q^{h}$ a risk-neutral probability modeling hedgeable risk (see Planchet et al. [2011] for the justification of this formula). Because of the decomposition

$$r_s(t) = r(t) + \omega(t)$$

we assume that we can split the probability $P^{nh}$ (which represents the risk associated with $\omega$) between two components, $P^{nh} = P^i \otimes P^{\omega}$.

In this decomposition, $P^i$ is associated with usual insurance risks, mostly mutualizable ones (mortality, structural ratchet, etc.) and $\omega$ stands for the risks associated with $\omega$.

We assume that insurance risks ($P^i$) and other risks ($P^{\omega} \otimes Q^{h}$) are independent.
2. Computing best estimates

The best estimate of the contract is

\[ BEL(0, T) = E^{P^O\otimes Q^E}(BEL^F(0, T)) \]

with \( BEL^F(0, T) = E^{P^F}(\Lambda|F) \). Conditioning to \( F \) means conditioning to financial risk (that is risk that affects the return of the contract, hedgeable or not).

At this stage, we need to make an assumption about \( \omega \) and \( \mu \) to obtain an explicit formulae.

The process \( \omega \) is modeled by an Ornstein-Uhlenbeck process

\[ d\omega(t) = k \times (\omega^* - \omega(t))dt + \sigma_\omega dB(t) \]

The market often retains a target rate of revalorization close to the risk-free rate (TME, 10-year OAT, etc.); this fact motivates our choice. Moreover, the revalorization rate by the contract is determined by the return on assets (its expectation equals to the risk-free rate under a risk-neutral probability) and also smoothing mechanisms induced by accounting principles.
Assume now that the surrender $\mu$ is decomposed into the sum of a structural (idiosyncratic) and a cyclical component $\mu(u) = \mu_i(u) + \mu_c(\omega(u))$.

Under these hypothesis, it can been shown that the best estimate is

$$\Lambda_x = \sum_{t=1}^{\infty} \frac{1}{(1+i)^t} F[x\alpha](T_x)$$

$S_i$ is given, and $\theta_1$ and $\theta_2$ have explicit form. In applications we will use the discrete form

$$BEL(0, T) = VR(0) \times \left[ \int_0^T S_i(t) \left( \mu_i(t) \times \theta_1(t) + \theta_2(t) \right) dt + S_i(T) \times \theta_1(T) \right]$$

$$BEL(0, T) = PM(0) \times \left[ \sum_{u=1}^{T} \frac{l_{u-1}}{l_0} \times \left( q_{u-1} \times \theta_1(\omega(0), u) + \theta_2(\omega(0), u) \right) + \frac{l_T}{l_0} \times \theta_1(\omega(0), T) \right]$$

We can now use this expression to project the balance sheet.
1. Simulating Best Estimates
2. Computing Best Estimates
3. Application: Balance Sheet Projection
3. Balance sheet projection

Balance sheet projection is crucial for calculations in the ORSA framework. To achieve this projection we need to compute reserves in an efficient way.

Here we use the formula set out above to calculate the best estimate from the mathematical reserve. For this we use the Markovian character of $\omega$. Then at time $t$ we have

$$
BEL(t, T) = \rho(t, T) \times PM(t)
$$

$$
\rho(t, T) = \int_{t}^{T} S_{i,i}(u) \left( \mu(u) \times \theta_{1}(\omega(t), u-t) + \theta_{2}(\omega(t), u-t) \right) du + S_{i,i}(T) \times \theta_{1}(\omega(t), T-t)
$$

$$
\rho(t, T) = \rho(t, T, \omega(t)) = \sum_{u=t+1}^{T} \frac{I_{u-1}}{l_{t}} \times \left( q_{u-1} \times \theta_{1}(\omega(t), u-t) + \theta_{2}(\omega(t), u-t) \right)
$$

$$
+ \frac{I_{T}}{l_{t}} \times \theta_{1}(\omega(t), T-t)
$$

This is the key component of our balance sheet calculation.
3. Balance sheet projection

We have to choose the dynamics for the main risk factors (under the historical probability)

\[ dr(t) = k_r \times (r_\infty - r(t))dt + \sigma_r dB_r(t) \]

\[ dr_A(t) = \mu_A dt + \rho \sigma_A dB_r(t) + \sqrt{1 - \rho^2} \sigma_A dB_A(t) \]

\[ d\omega(t) = k_\omega \times (\omega_\infty - \omega(t))dt + \frac{\rho_s \sigma_\omega}{\sqrt{1 - \rho^2}} dB_A(t) + \sqrt{1 - \rho^2} \frac{\rho_s \sigma_\omega}{1 - \rho^2} dB_\omega(t) \]

and then describe the dynamics of the asset value \( A \) and cash flows of benefits \( F \):

\[ A(t + 1) = A(t) \exp\left(\mu_A + \rho \sigma_A (B_r(t + 1) - B_r(t)) + \sqrt{1 - \rho^2} \sigma_A (B_A(t + 1) - B_A(t))\right) - F(t + 1) \]
3. Balance sheet projection

One can show that

\[
F(t+1) \approx VR(t) \times \frac{l_{i,d}}{l_{i,0}} \times \exp\left(\sum_{u=0}^{t-1} \eta \omega(u)\right) \times \left(1 - (1-q_i(t)) \times e^{\eta \times \omega(t)}\right)
\]

\[
\approx PM(t) \times \left(1 - (1-q_i(t)) \times e^{\eta \times \omega(t)}\right)
\]

\[
A(t+1) \approx A(t) \times \exp\left(\mu_A + \rho \sigma_A \epsilon_r(t+1) + \sqrt{1-\rho^2} \sigma_A \epsilon_A(t+1)\right)
\]

\[- PM(t) \times \left(1 - (1-q_i(t)) \times e^{\eta \times \omega(t)}\right)
\]

\[
PM(t+1) = PM(t) \times \exp\left(\int_t^{t+1} (r(u) + \omega(u) - \mu(u)) du\right)
\]

\[
\approx PM(t) \times \exp\left(r(t) + (1+\eta) \omega(t)\right) \times (1-q_i(t)))
\]
3. Balance sheet projection

The own fund have the following expression

$$E_{t+1} = e^{r(t)} \times \left\{ A(t) \times e^{\mu_A \times r(t) + \rho \sigma_s \times (t+1) + \sqrt{1 - \rho^2} \sigma_s \times (t+1)} - PM(t) \times e^{-r(t)} \times \left(1 - \left(1 - q_i(t)\right) \times e^{\eta \times \omega(t)}\right) - PM(t) \times e^{(1+\eta) \omega(t)} \times \left(1 - q_i(t)\right) \times \rho \left(t + 1, T, \omega(t + 1)\right)\right\}$$

and we deduce that the SCR is (see Guibert et al. [2014] for a discussion about SCR calculations)

$$SCR_t = E_t - VaR_t \left\{ E_{t+1} \times e^{-\int_{0}^{t+1} r(u) du} ; 0.5\% \right\}$$

$$SCR_t \approx E_t - VaR_t \left\{ A(t) \times e^{\mu_A \times r(t) + \rho \sigma_s \times (t+1) + \sqrt{1 - \rho^2} \sigma_s \times (t+1)} - PM(t) \times e^{-r(t)} \times \left(1 - \left(1 - q_i(t)\right) \times e^{\eta \times \omega(t)}\right) - PM(t) \times e^{(1+\eta) \omega(t)} \times \left(1 - q_i(t)\right) \times \rho \left(t + 1, T, \omega(t + 1)\right)\right\}$$
3. Balance sheet projection

On this basis the projection of the following variables can be performed over the next 5 years:

- the value of the mathematical reserve;
- the benefit stream;
- the market value of the assets;
- the simulated paths of financials variables (this information is required for the ORSA process).

At last, this allows projecting the evolution over the next 5 years of the net asset value (NAV) represented via the following graph. This is a very useful ORSA indicator.
« Pillar one » techniques need to be carefully set-up to be efficient. They are not suited to project the balance sheet. Having a closed formula to go from the mathematical reserve to the best estimate evaluation of the reserve improves dramatically the performance of calculations. Being easily reproducible, it facilitates the process of audit and control.

Such models can be built analyzing the main risks of the contracts. For example, observing that a (French) saving contract is mainly non-hedgeable, because of the accounting rules effect on the revalorization rate of the contract help building an appropriate calculation framework.

A similar approach can be used for pensions, see Bonnin et al. [2014a].
This approach also provides us a powerful tool for making projections of SCR along a « critical path ». This is especially interesting when seen as part of an ORSA process, like time dependent stress scenario analysis (cf. Guibert et al. [2014]).

This first analytical framework can then be expanded to capture more complex effects, such as the wealth effect of the insurer through its management of unrealized losses.

This will be the subject of future work to jointly model the book value and market value of assets.


LAIDI Y. [2013] Problématiques de calibration en vue de l’évaluation des risques de taux, de défaut et de liquidité, Mémoire d’actuaire, CNAM.


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