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INVARIANT SUBSPACES: AN ALTERNATIVE FOR INTRODUCING EIGENVECTORS AND EIGENVALUES

Abstract. The concepts of eigenvalue and eigenvector are typically approached algorithmically in introductory linear algebra courses. However, a more conceptual orientation involves connecting these notions to the concept of one-dimensional invariant subspace, which allows for the introduction of eigenvectors prior to eigenvalues. In this study, we present data collected from interviews with two linear algebra instructors as they worked with a specific linear transformation in both paper-and-pencil and dynamic geometry environments. The data were analyzed using the perspectives of APOS theory and the theory of Mathematical Working Spaces in a complementary manner. The results indicate that dynamic representations facilitate the establishment of relationships between eigenvectors, eigenvalues, and invariant subspaces. This approach proves to have potential for developing a deeper understanding of the related concepts.

Keywords. Invariant subspace, eigenvector, eigenvalue, dynamic geometry, linear algebra.

Résumé. Les concepts de valeur propre et de vecteur propre sont généralement abordés de manière algorithmique dans les cours d'algèbre linéaire de niveau introductif. Cependant, une orientation plus conceptuelle consiste à relier ces notions au concept de sous-espace invariant unidimensionnel, ce qui permet d'introduire les vecteurs propres avant les valeurs propres. Dans cette étude, nous présentons des données collectées lors d'entretiens avec deux enseignants d'algèbre linéaire qui ont travaillé avec une transformation linéaire spécifique dans des environnements tant papier-crayon que de géométrie dynamique. Les données ont été analysées en utilisant les perspectives de la théorie APOS et de la théorie des espaces de travail mathématique de manière complémentaire. Les résultats indiquent que les représentations dynamiques facilitent l'établissement de relations entre les vecteurs propres, les valeurs propres et les sous-espaces invariants. Cette approche a le potentiel de développer une compréhension plus approfondie des concepts liés.

Mots-clés. Sous-espace invariant, vecteur propre, valeur propre, géométrie dynamique, algèbre linéaire.

Eigenvalues and eigenvectors are usually the last topic covered in an introductory linear algebra course. The understanding of these notions is enriched if explicit connections are made with concepts such as linear transformation, vector space, subspace, basis and linear independence. Dorier (2000) reports that towards the end of a linear algebra course, instructors rush through the topic of eigenvalues in an algorithmic manner. Perhaps partially for this reason Lapp et al. (2010) found that in

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the context of concept maps, for most students the only other expression involved in the eigenvalue-eigenvector clump (cluster) was the expression 'matrix diagonalization' and the whole clump was only loosely related to others. All the other clusters included many more concepts and were connected more closely with other clusters, including linear independence and linear transformation. This shows that in general students do not associate eigenvectors with other important concepts of linear algebra and that there is need for instructional strategies to help students make connections within linear algebra theory.

Inspired by Sierpinska's following quote, in this paper we explore a theoretical oriented approach that can help students make sense of eigenvectors/eigenvalues and also establish connections within linear algebra:

> [I]n the structural mode, the notion of eigenvalue cannot be reduced anymore to that of a root of a polynomial. It must be thought of as a scalar related to invariant one-dimensional subspaces of a linear operator. It is an object of reflection and a concept; not an outcome of a calculation. (Sierpinska, 2000, p. 236)

The aim of this research study is to inquire about mental constructions that some university instructors, as individuals with advanced knowledge of linear algebra, make use of, when relating the concepts of invariant subspace, eigenvector and eigenvalue. We point out that it is not our intention to delve into the construction of the concepts but to explore how individuals understand their relationships. In the remaining of this paper, we first explain the mathematical notions in question and the relationships between them. After that we offer a non-exhaustive literature review with the aspects that relate to this study. We then present aims of our research and the theoretical approach that guided it. The way in which the use of dynamic geometry was conceived within the context of our investigation, and the method employed follow. In the results section, we analyze the empirical data; following that, we offer a discussion on our findings.

1. Mathematical aspects, invariant subspaces, eigenvectors and eigenvalues

The study of relationships between invariant subspaces, eigenvectors and eigenvalues can be useful in constructing an image for the existence of infinitely many eigenvectors associated with a single eigenvalue, which poses a difficulty for students (Caglayan, 2015; Wawro et al., 2019). It can also aid in the understanding of 0 as an eigenvalue, which is problematic according to Soto and García (2002). Furthermore, with this approach the notion of eigenvalue is not favored over the notion of eigenvector as is the case in the algorithmic approach where calculation of the roots of the characteristic polynomial is required before obtaining eigenvectors. Rasmussen and Keynes (2003) suggest that introducing eigenvectors first as an instructional strategy is more in line with students' reasoning as opposed to the traditional way of presenting eigenvalues first and then calculating eigenvectors. We now present the definitions of some of the mathematical notions involved in our study as well as some related results.

Let's suppose that $\mathscr{L}(V)$ is the set of all linear transformations of the vector space V on itself. An invariant subspace is defined as follows:

Definition: "Suppose $T \in \mathscr{L}(V)$. A subspace U of V is called *invariant* under T if $u \in U$ implies $T(u) \in U$" (Axler, 2015, p. 132); that is, $T(U) \subseteq U$. It follows that V and $\{0\}$ are always invariant subspaces.

The importance of this definition lies in the fact that invariance of a subspace U implies that, the restriction of the linear transformation to U is also a linear transformation on that smaller domain. This becomes a useful tool when considering the decomposition of the vector space V to its direct sum of subspaces, each of which is invariant, so that studying the effect of the linear transformation on each subspace – which is a lot easier – gives us information about the effect of that linear transformation on the whole space (Axler, 2015).

Now, let's consider the following one-dimensional subspace U of V, for a vector v in V with $v \neq 0$: $U = \{\lambda v \mid \lambda \in F\} = \langle v \rangle$, which denotes all the multiples of v, where λ is a scalar that belongs to a field F. If U is a T-invariant subspace, it has to satisfy the condition $T(w) \in U$ for all $w \in U$. In particular, for v in U, $T(v) = \lambda v$ for some $\lambda \in F$. Conversely, if v is a vector in V such that $T(v) = \lambda v$ for some $\lambda \in F$, then $\langle v \rangle$ is a T-invariant subspace of dimension 1. This reasoning reveals an intimate connection between the equation $T(v) = \lambda v$ and the invariant subspaces of dimension 1. Hence, we can see that the concepts of eigenvector and eigenvalue can be motivated by studying invariant subspaces.

Next, let's remember the definitions of eigenvector and eigenvalue as they are commonly presented in linear algebra courses.

Definition (eigenvalue). "Suppose that $T \in \mathscr{L}(V)$. A number $\lambda \in F$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $T(v) = \lambda v$" (Axler 2015, p. 134).

Definition (eigenvector). "Suppose that $T \in \mathscr{L}(V)$ and $\lambda \in F$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $T(v) = \lambda v$" (Axler 2015, p. 134).

The above definitions imply that V has a one-dimensional invariant subspace under T if and only if T has an eigenvalue (Axler, 2015, p. 134). In particular, if v in V is an eigenvector of T then $\langle v \rangle$ is a one-dimensional invariant subspace of V under T with the property that every element of $\langle v \rangle$ is an eigenvector of T. On the other hand, if $\langle v \rangle$ is a T-invariant subspace then there exists a $\lambda \in F$ such that $T(w) = \lambda w$ for all $w \in \langle v \rangle$.

In the two-dimensional vector space $\mathbf{R}^2$ these relationships and properties can be visualized through the representations of Analytic Vector Geometry. The eigenvectors of T in $\mathbf{R}^2$ either preserve direction under the transformation, $T(v) = \lambda v$ dilates or compresses v by a factor of λ its associated eigenvalue; or they can reverse direction if λ is negative, again dilating or compressing by a factor of λ (Figure 1).

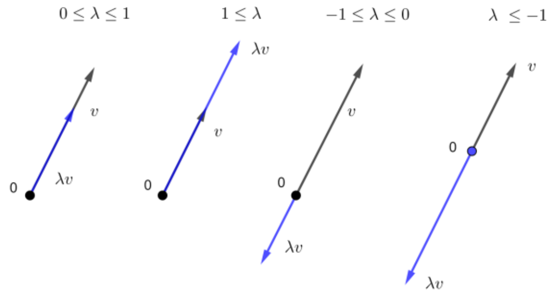

Figure 1. Geometric representation of an eigenvector in $\mathbf{R}^2$ (adapted from Anton and Rorres, 2014, p. 292).

In the two-dimensional vector space $\mathbf{R}^2$ these relationships and properties can be visualized through the representations of Analytic Vector Geometry. One-dimensional invariant subspaces can be represented as lines passing through the origin and corresponding to sets of eigenvectors associated with an eigenvalue λ. We consider that this kind of visualization not only allows to "see" what algebraic symbols denote, but also they favor reflections that are less accessible in an algebraic context. In that sense, work with algebraic and geometric representations complement each other.

2. Literature review

Approaches to teaching eigenvectors and eigenvalues differ. Some researchers find motivation in application tasks such as cryptography (Siap, 2008), while others use dynamic geometry for enhancing visualization and focusing on properties as well as working with different representations (Gol Tabaghi & Sinclair, 2013; Caglayan, 2015). Plaxco et al. (2018) utilize the notions of 'stretch factors' and 'stretch directions' of a linear transformation in a geometric context, building on students' mathematical activity.

Targeting a conceptual understanding of students through an inquiry-oriented approach, Bouhjar et al. (2018) argue for the need to establish connections between procedures such as finding the roots of the characteristic polynomial, solving the equation $Ax = \lambda x$ and the reasons why these procedures work, as well as their relationship with the definitions involved. Understanding of the properties of linear combinations of eigenvectors and eigenspace was studied by Wawro et al. (2019);

the authors suggest that the study of eigenspaces as subspaces can enhance the connections that students make among the related concepts.

Sinclair and Gol Tabaghi (2010) report how mathematicians think about eigenvectors in informal ways. Some of them visualize a linear transformation modifying the space in a certain manner, and the eigenvectors are those that do not change direction or lie in the opposite direction under the transformation.

From an APOS (Action – Process – Object – Schema) perspective, Salgado and Trigueros (2015) propose a genetic decomposition for eigenvalue, eigenvector and eigenspace in a modelling context, which they report was useful in helping students construct an Object conception of these concepts. Betancur et al. (2022) propose another genetic decomposition for eigenvalue and eigenvector that considers geometric aspects.

We did not find any study that examines the understanding of eigenvectors and eigenvalues in relation to invariant subspaces. Hence we set out to investigate this connection; we decided to work with instructors instead of students, because of their background and knowledge about linear algebra topics, since it is not likely that undergraduates have had the experience of being introduced to invariant subspaces.

3. Aims of the study

The aim of this study is to explore linear algebra instructors' conceptions about the notions of eigenvector and eigenvalue in relation to invariant subspaces. As mentioned before, connections between these concepts are not normally presented in an introductory linear algebra course. We wanted to observe how instructors reason about and make sense of these notions, as well as the difficulties that they might experience. The purpose of this investigation was two-fold. On the one hand, in the absence of previous research on this theme, we wanted to establish certain theoretical elements for the construction of these concepts. On the other hand, we were interested in coming up with pedagogical suggestions for the improvement of the learning process in linear algebra.

4. Theoretical framework

Our study makes use of two theoretical approaches in a complementary manner. The first one, APOS theory, provides the necessary tools to model individuals' construction of concepts (Arnon et al., 2014). Through the second one, Mathematical Working Spaces (MWS), connections between an epistemological plane associated to mathematical content, and a cognitive plane associated to an individual involved in mathematical activity, can be analyzed (Kuzniak et al., 2016).

According to APOS Theory, the construction of knowledge passes through stages known as *mental structures*; transition between these stages occurs by means of *mental mechanisms*. By custom, the first letter of each mental structure is written in a capital letter, to differentiate these conceptions from their other meanings. *Actions* are the first building blocks of knowledge construction which are driven externally through mathematical formulas or step-by-step procedures over which the individual does not have internal control. As the person repeats and reflects on them, they are *interiorized* into Processes where the individual can perform the same Actions in her/his mind without the need for external stimuli.

Another way to build new processes from existing ones is through the *reversion* mechanism. If the individual has constructed a Process conception, internalized Actions can be executed on an object that we can consider as an input object, that is transformed into an output object. Reversion consists of recovering the input object from the output object of the original process. Another mechanism, namely *coordination*, allows bringing together two processes in a way to produce a new process with its own characteristics.

The construction of an *Object* conception is characterized by the ability to apply Actions on the Processes which are then *encapsulated*. The collection of different Actions, Processes and Objects related to a concept and their connections gives rise to a *Schema*. The coherence of a Schema is determined by the individual's ability to evoke it in different situations where it can be useful. Knowledge construction continues in a spiral manner, each newly constructed Object being subject to the application of new actions on them, and so forth.

The MWS theory conceives mathematical work as a gradual process that evolves with the interactions between the epistemological elements of mathematical contents and the cognitive processes of individuals solving tasks (Kuzniak, 2022). At the epistemological plane, in one of the three components we recognize a set of tangible signs called representamens such as geometric images, algebraic symbols or graphs; a second component includes a set of material artifacts such as drawing instruments or dynamic mathematics software, as well as symbolic artifacts such as algorithms for solving systems of equations. The third component refers to a theoretical referential integrated by a set of axioms, definitions, properties and theorems organized to support arguments and demonstrations. The cognitive plane, on the other hand, is composed of three thought processes. The process of visualization associated to the identification and interpretation of signs, the construction process related to artifacts and their techniques of use, and a discursive process of proof that operationalizes the elements of the theoretical referential (Kuzniak et al., 2016).

Interactions between the epistemological and cognitive planes are explained through the semiotic genesis, the instrumental genesis, and the discursive genesis of proof. Semiotic genesis refers to the relationship between the representamen component and the visualization process. It explains the link between the syntactic aspect or that concerning the combination of signs and the semantic aspect (Kuzniak et al. 2016).

Instrumental genesis relates artifacts to the construction process by explaining the actions a user takes to appropriate techniques to manipulate an artifact as well as the choice of an artifact to perform intentional actions to solve a task (Kuzniak, 2022). The discursive genesis links the theoretical referential to the proof; this genesis is manifested by the elaboration of deductive discourses supported by the content of the referential or by identifying and including new properties in the referential as a product of instrumental treatments or visualizations (Kuzniak, 2022).

A possible circulation of knowledge is conceived through the activation of vertical planes that connect the epistemological and cognitive planes. Figure 2 shows a diagram used to visualize the development of mathematical work by means of a circulation between vertical planes. The plane defined by semiotic genesis and discursive genesis is the Semiotic-Discursive plane; the plane determined by instrumental genesis and discursive genesis is the Instrumental-Discursive plane. And the plane determined by the semiotic genesis and the instrumental genesis is the Semiotic-Instrumental plane. In APOS theory the role that the geneses play is not explicit; in this study we are interested in particular in the semiotic and instrumental geneses.

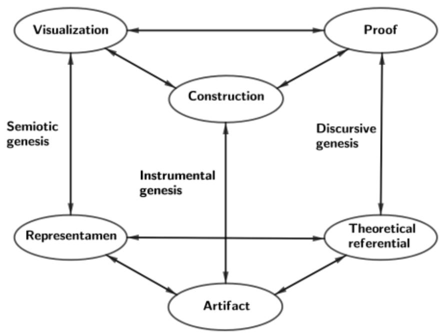

Figure 2. Diagram of the abstract structure of the MWS (adapted from Kuzniak and Richard, 2014)

Mathematical work often involves interaction between two or more branches or fields of mathematics. Within the theory of Mathematical Working Spaces, Montoya and Vivier (2014) propose the notion of *mathematical domain* defined through its own objects together with their representations, as well as by its own theoretical referential in which the elements of the epistemological plane are identified. To be considered a domain, it is crucial that the mathematical community recognize it as such (Montoya and Vivier, 2014). Because of their breadth, subdomains can be considered, for example, Linear Algebra as a subdomain of Algebra.

We consider that the MWS perspective offers a potential explanation for how interactions with signs and technological artifacts can assist individuals in organizing and developing mental structures that aid in solving problems. Additionally, APOS theory includes elements not considered in the cognitive plane of MWS, such as the analysis of mental constructions involved in the genesis activation or domain changes. The heuristic strategy called *combination* is used to identify the dialogue between proposed theories, according to the spectrum of connecting theories of Prediger et al. (2008).

5. Use of dynamic geometry

The concepts of eigenvector and eigenvalue lend themselves easily to an exploration by means of the tools of dynamic geometry in 2-dimensions. Dragging of vectors, lines and other geometrical shapes can help visualize the relationships these objects hold, as well as aid in making connections between different representations. For example, the relationship between eigenvectors of a linear transformation associated with one eigenvalue can be visualized (Gol Tabaghi, 2014). Presenting linear algebra concepts using dynamic geometry also makes it possible to introduce eigenvectors before eigenvalues.

Sierpinska et al. (1999) designed a dynamical geometry environment in which the two-dimensional vector space was represented by a geometric model without coordinates with the intention to enhance students' thinking about vectors, linear transformations, eigenvectors and other linear algebra concepts with the intention to avoid the obstacle of formalism (Dorier et al., 2000). Gol Tabaghi and Sinclair (2013) presented some students with the task of finding eigenvalues and eigenvectors of matrices in a dynamic geometry environment. They report that participating students developed a synthetic-geometric mode of thinking (Sierpinska, 2000) which showed dynamic characteristics.

For this study we designed a dynamic geometry environment, with the purpose of investigating its effect on the mental constructions that individuals make while observing their mathematical work. Figure 3 shows the initial screen in GeoGebra software. The little boxes are labeled: image of vector *v*; trace of image vector; free line; line through origin; image of the line; circumference; norm of *v* and norm of the image of *v*.

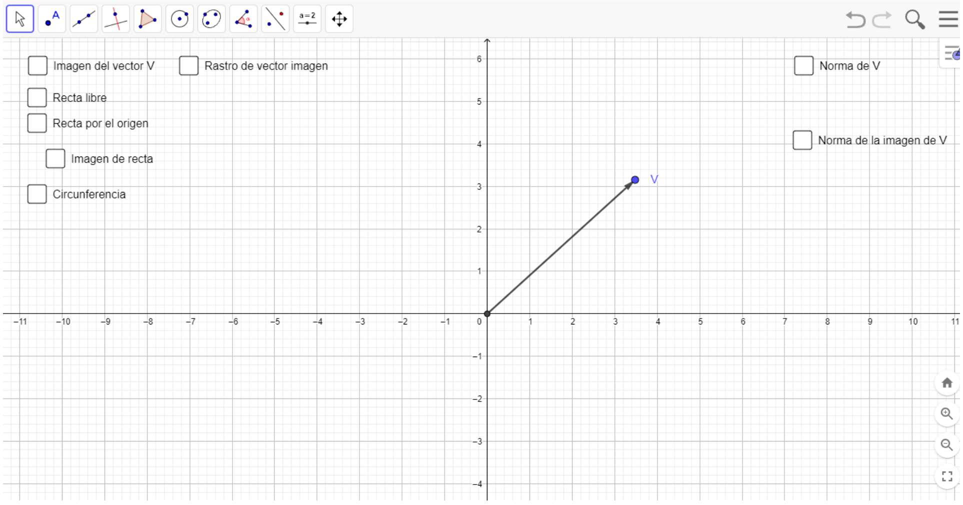

Figure 3. Initial GeoGebra screen

The user has four options for the dragging of the movable vector: all over the screen; along a movable red line that passes through the origin once the corresponding box is activated; along a movable red line not necessarily passing through the origin once the corresponding box is activated; and along a circumference with center at the origin and variable radius once the corresponding box is activated. Figure 4 shows two examples of possible situations.

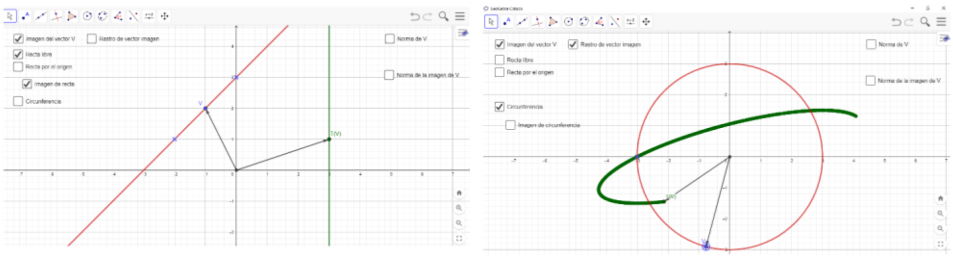

Figure 4. GeoGebra screen with: a) movable vector along a movable line that does not pass through the origin and the image of the movable line; b) image of vector *v*, trace of the image vector and circumference boxes activated

We note that in this environment it is possible to observe that the image of any vector of an invariant subspace lies in the same direction or the opposite direction with respect to the original vector, or it can be the zero vector. We also observe that all vectors within the image of an invariant subspace share a common scaling factor when compared to their corresponding preimages. This observation serves as a compelling rationale for exploring the concept of eigenvalues. Moreover, this approach accommodates non-invertible linear transformations and enhances our comprehension of zero as an eigenvalue.

6. Method

For the purposes of this paper and because of the absence of previous research with focus on the learning of eigenvalues and eigenvectors motivated by their relationships with one-dimensional invariant subspaces, we adapted the first component of the APOS theory research cycle: theoretical analysis. Commonly, theoretical analysis is informed by previous research, analysis of textbooks, and practical experience of the researchers. Because of the lack of these elements, we decided to elaborate an exploratory questionnaire, as a data collection instrument, to gather information for the future design of a viable cognitive model.

We present semi-structured interviews performed with two experienced linear algebra instructors who work at a large public university in Mexico. One of them holds a Ph.D. and the other a master's degree in mathematics; both of them have over ten years of experience in teaching mathematics courses in mathematics and physics undergraduate programs.

Both instructors mentioned that they know the functioning of dynamic geometry and GeoGebra, but they never use it. During the interview they were explained about the GeoGebra environment designed for exploring a linear transformation.

The interview with the first instructor consisted of three sections, the first one of which focused on the concept of invariant subspaces and contained the following questions:

- Can you explain what an invariant subspace is?
- How would you determine the invariant subspaces of a linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$?
- Can you explain graphically what it means to find the invariant subspaces of a linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$? and
- Consider the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$? such that $T(x,y) = (-x + y, \frac{1}{2}y)$. How would you determine the invariant subspaces of T ?

The second part was guided by the following questions and focused on the concepts of eigenvector and eigenvalue as well as their relationship with invariant subspaces.

- How do you determine the eigenvectors and eigenvalues of a linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$?
- Can you give a graphical interpretation of the eigenvectors and eigenvalues of a linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$?

- Consider the linear transformation T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$? such that $T(x,y) = (-x + y, \frac{1}{2}y)$. Can you find the eigenvectors and eigenvalues of T ? Can you interpret graphically the eigenvectors and eigenvalues of this transformation?
- Do you observe any relationship between invariant subspaces, eigenvectors and eigenvalues of the linear transformation $T(x,y) = (-x + y, \frac{1}{2}y)$?

The last part of the interview involved work with dynamical geometry where the instructor was presented with the task of exploring the linear transformation $T(x, y) = (-x + y, \frac{1}{2}y)$ in the previously designed GeoGebra environment. The following two questions were used as a guide:

- Can you find the invariant subspaces of $\mathbf{R}^2$ under this transformation?
- Can you find the eigenvectors and eigenvalues of this transformation?

After analyzing the first instructors' interview and realizing that he was relying heavily on his work in the paper-and-pencil section to answer the questions in the dynamic geometry context, we decided to change the order of the sections. After asking the second instructor an explanation of the invariant subspace concept, the interview continued with work in the GeoGebra environment, using the same linear transformation as in the first interview, but this time the second instructor was not provided with the algebraic expression. He was asked to explore the invariant subspaces as well as the eigenvectors and eigenvalues associated with the transformation. After he completed this part, he was asked to find an algebraic expression for the transformation that he had just explored in the dynamic geometry environment, as well as to determine the eigenvalues and eigenvectors of the linear transformation, using the algebraic expression that he had come up with.

Both interviews were videotaped and subsequently transcribed. In line with the methodology associated with APOS theory (Arnon et al., 2014) the transcriptions were analyzed by both researchers independently; observations were compared until an agreement was reached about the interpretations from our theoretical lens. The use of MWS fitted perfectly into this methodological approach.

7. Results

7.1. The first instructor

The first instructor (we will call him by the pseudonym Daniel (D)), as we mentioned before, first worked on the problem using paper and pencil, and then explored it in the dynamic geometry environment. Although he initially used some features of the concepts of vector subspace and invariant subspace by resorting to geometric representations, his reflections mainly focused on using algorithms and algebraic procedures.

7.1.1. The subspaces of $\mathbf{R}^2$

When thinking about linear algebra from its most abstract aspects, the use of geometric representations may seem unnecessary—and even an obstacle—to generalization, which is one of its main objectives. However, using geometric representations—visual or mental—can favor reasoning that may prove to be useful in solving problems posed in an abstract context; as we mentioned earlier, some mathematicians resort to this approach (Sinclair & Gol Tabaghi, 2010).

Daniel's initial strategy to determine the invariant subspaces of the linear transformation $T(x, y) = (-x + y, \frac{1}{2}y)$ focused on using the definition by evaluating T in a generic element of a generic subspace of $\mathbf{R}^2$. He explained that if (x, y) is an element of a subspace H and its image belongs to H, then the subspace is invariant. However, when Daniel was asked to list the invariant subspaces of $\mathbf{R}^2$ under this transformation, he initially had difficulty finding them.

> D: I think in this case it would be easier to decide if you give me an arbitrary subspace, the one you want, to decide if it is T-invariant. It's easier in this case than the problem here. Because describing all subspaces is a problem: how far do I go? How far do you want to describe?

It is true that there is no established general algorithm to obtain the invariant subspaces of a vector space under a given transformation. To find them, the instructor had to mobilize various elements of his theoretical referential and use the existing relationships in his thinking between his conceptions of subspace, linear transformation and invariant subspace. His reflection about the geometric aspects of a subspace of the two-dimensional vector space led him to a reasoning that moved between different representations and helped link algebraic and geometric aspects. After thinking for a few minutes, Daniel explained:

> D: Who are the subspaces of $\mathbf{R}^2$? H is a subspace if it is a line through the origin, right? They are (x, kx) with k a real number. But what description would H have? It is the set of (x, y)..., where $y = kx$. They are the (x, kx) with k real number.

The instructor referred to a non-trivial subspace as $H = \{(x, y) \mid y = kx\}$ and performed treatments on the expression to conclude that this subspace can also be represented as $\langle\{(1, k)\}\rangle$, as shown in Figure 5. Daniel indicated that the subspaces "are the straight lines of slope k through the origin", emphasizing the importance of the position vector $(1, k)$ as the generating vector.

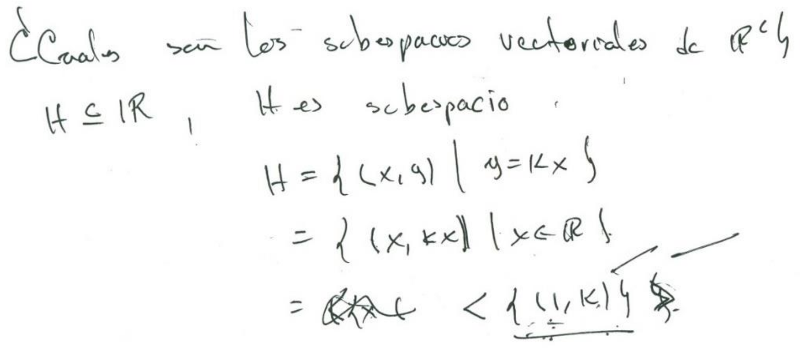

Figure 5. Algebraic description of subspaces elaborated by Daniel

It can be observed from the perspective of APOS theory that Daniel used his Process conception of subspace to validate his conjecture that the straight lines passing through the origin are subspaces. The instructor also resorted to his Process conception of span to recognize a straight line passing through the origin as a subspace with a generator. Using his Process and Object conceptions of a line, he represented a generic line passing through the origin, with the equation $y = kx$. He manipulated this expression and, returning to the underlying process, he explained that the line could also be represented by the set notation $\{(x, kx) \mid k \in \mathbf{R}\}$.

To recognize this set as a subspace, Daniel applied his Process conception of span to the set $\{(x, kx) \mid k \in \mathbf{R}\}$ and expressed it as $\langle\{(1, k)\}\rangle$. The notation selected by the instructor helped identify the line through the origin as the subspace generated by $(1, k)$. The dynamism of the Processes was evident in every representation; the instructor could imagine each of the elements of any given line and represented them as the set $\{(x, kx) \mid x \in \mathbf{R}\}$; he also thought of them as multiples of $(1, k)$.

Daniel's mathematical work was developed as an interchange between the domain of Analytic Geometry and the domain of Linear Algebra. He resorted to the Cartesian plane as a representation of $\mathbf{R}^2$. He did this not only for changing the register that would make it easier for him to express the vector space, but he deliberately used the slope of a straight line passing through the origin to refer to the subspaces of $\mathbf{R}^2$. The slope k of the line $y = kx$, which is an element of Analytic Geometry, served to move further with the description of subspaces as sets of the form $\langle\{(1, k)\}\rangle$.

On the other hand, his mathematical work was situated on the Semiotic-Discursive plane. The discursive genesis was activated when, with the theoretical tools of his referential of Analytic Geometry and Linear Algebra, Daniel established connections between the straight lines passing through the origin and the vector subspaces of $\mathbf{R}^2$. Simultaneously Daniel activated the semiotic genesis by encoding, employing the algebraic register, and the graphic characteristics of a subspace.

Through treatments in the algebraic register, and only after pointing out that a generic vector of the subspace is of the form (x, kx), Daniel visualized features that he had not considered before. As a result of his work in the semiotic genesis, he recognized the importance of the parameter k to identify a subspace when representing it as a straight line in the plane, where the slope is a crucial element in determining its position.

7.1.2. Determination of subspaces using the dynamic environment

After formulating an expression for a generic subspace of $\mathbf{R}^2$ from the equation of a line passing through the origin, Daniel failed to determine the invariant subspaces using algebraic techniques. The instructor equated the image of any vector $T(x, kx) = ((k - 1)\, x, \frac{1}{2}\, kx)$ with a vector of the form (y, ky) to solve the resulting system of equations. However, he did not achieve appropriate results because he focused on obtaining values for x and y instead of for the parameter k. Obtaining the values of k from the system would have allowed him to determine a generator vector and, consequently, a subspace with the desired characteristics.

Daniel continued analyzing the linear transformation by obtaining an associated matrix and calculating its eigenvalues, eigenvectors and eigenspaces using conventional algorithms. He explained that the eigenvalues of T are $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{2}$, and the respective eigenspaces are the X-axis or $\langle\{(1, 0)\}\rangle$ and the line with equation $y = \frac{3}{2}x$, as shown in Figure 6.

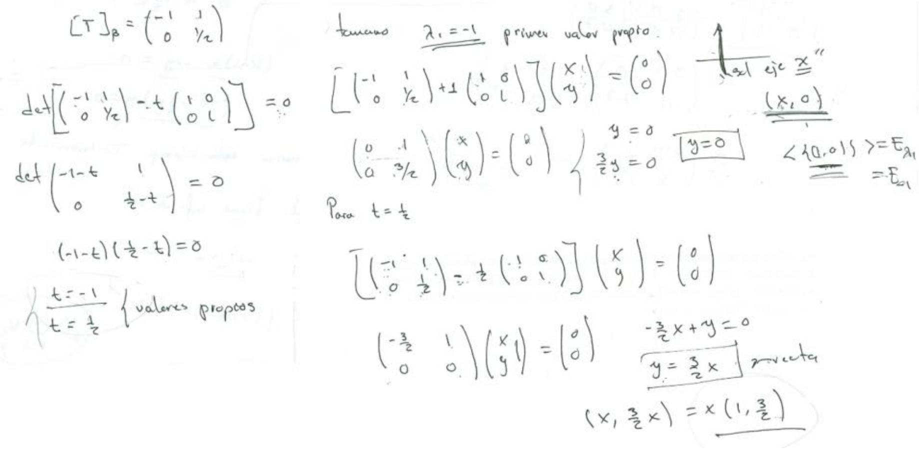

Figure 6. Calculation of eigenvalues and eigenspace performed by Daniel

To inquire into his conceptions from a graphical perspective, we proposed him the use of the dynamical environment to determine the invariant subspaces. Before starting work with GeoGebra, Daniel had a few minutes of reflection and afterwards explained the following:

D: A subspace is invariant if the transformation sends the line to itself. It should be all those straight lines, I think, so that *T* of the straight line is on the straight line.

Daniel started manipulating the environment by checking the boxes 'line through origin' and 'image of the line'. Still, he did not activate 'image of vector *v*'. He rotated the mobile straight line around the plane's origin until it coincided with its image and said that "you would just go around and see where they coincide". Using his knowledge of Analytic Geometry, the instructor obtained the equation $y = \frac{3}{2}x$ as the equation of the first invariant subspace that he detected. He also identified the line $y = 0$ as an invariant subspace following the same circular dragging technique.

During the exploration of the dynamic environment, Daniel's mathematical work was situated on the Semiotic-Discursive plane. He activated the discursive genesis when he reasoned about vector subspaces of $\mathbf{R}^2$ and about what must happen for a subspace to be invariant. Daniel thought of a graphical representation for the subspaces of $\mathbf{R}^2$ and mentally applied the linear transformation to that subspace. He reasoned about its image in terms of the selected representation. Although the representamen straight line is "shared" by the Analytic Geometry and Linear Algebra domains, the instructor's discourse exhibited its use associated with a Linear Algebra concept.

Afterwards, the mathematical work was placed on the Semiotic-Instrumental plane. Daniel generated usage schemes in the digital environment by rotating the mobile straight line to track the subspaces that met the proposed condition. We recognize in these actions a process of instrumentalization since by rotating the mobile line until it coincided with its image, the instructor brought the construction that he had elaborated mentally beforehand to the digital medium.

From the viewpoint of APOS theory, we interpret that Daniel performed an Action on the Object vector subspace in the dynamic environment when he rotated the moving line as a graphical representation of that space. By turning the line, he applied the Action linear transformation to the Object subspace and detected the lines whose images are the lines themselves.

7.2. The second instructor

The second instructor (we will call him with the pseudonym Fernán (F)) worked from the start, with the dynamic environment designed with GeoGebra exploring the graphical aspects of the linear transformation in the plane, understanding it as a representation of the vector space $\mathbf{R}^2$. He didn't know what the rule of the transformation that he was exploring was, nor its associated matrix. He just had access to its graphical aspects through the manipulation of the dynamic environment prepared in GeoGebra.

7.2.1. The graphical determination of invariant subspaces

To address the concept of invariant subspace of a vector space under a linear transformation, Fernán started with the recognition of the possible subspaces of $\mathbf{R}^2$, referring to them as "the zero or straight lines passing through the origin". He explained that an invariant subspace "is a subspace that does not change when a linear transformation is applied to it. It does not stay in the same way, but as a set, it remains the same". He wrote the definition and gave an example as it is shown in Figure 7 and explained further how he thinks about an invariant subspace. We note that his conception of an invariant subspace staying the same as a set under a linear transformation is not coherent with the definition that he wrote, since this last one implies that the image of an invariant subspace is contained in the subspace itself, but these two sets do not have to be the same.

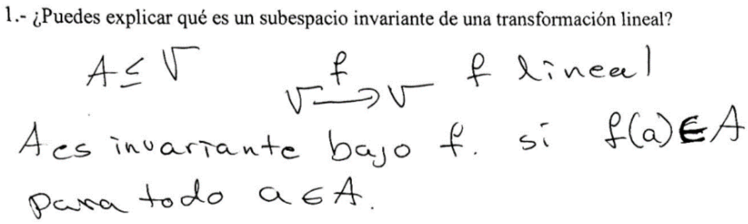

Figure 7. Definition of invariant subspace by the instructor Fernán

F: It is possible that [the transformation] rotate the straight line, but as a set it remains being the straight line. The order of the elements maybe not the same, but they remain in the same space.

Because of this conception that an invariant subspace stays the same under a linear transformation, Fernán did not consider the existence of one-dimensional invariant subspaces whose image is the zero vector; nor he referred to $\mathbf{R}^2$, but he did consider {0} as an invariant subspace. The instructor started the dynamic exploration of the linear transformation by activating the 'image of vector *v*' box in the GeoGebra environment. He dragged the mobile vector from the position (1, 0) through the first quadrant and asserted that invariant subspaces different from {0} did not exist.

Fernán explained that to identify invariant subspaces, it is necessary to find a basis of $\mathbf{R}^2$ such that "some of its elements remain, I mean, it's going to go to itself"; meaning that the image of an element is itself. For this reason, in his initial exploration, he examined the images of the standard basis vectors and asserted the nonexistence of invariant subspaces. In a second exploration, he slowly rotated the mobile vector in counterclockwise direction until it became collinear with its image in the first quadrant of the plane, as shown in Figure 8, and declared the following:

F: Yes, here the vectors are parallel, v and $T(v)$ are parallel; this means that the space generated by v when applying T, which is this line, will remain invariant when T is applied to it. (He draws with his finger on the screen the line that passes through v).

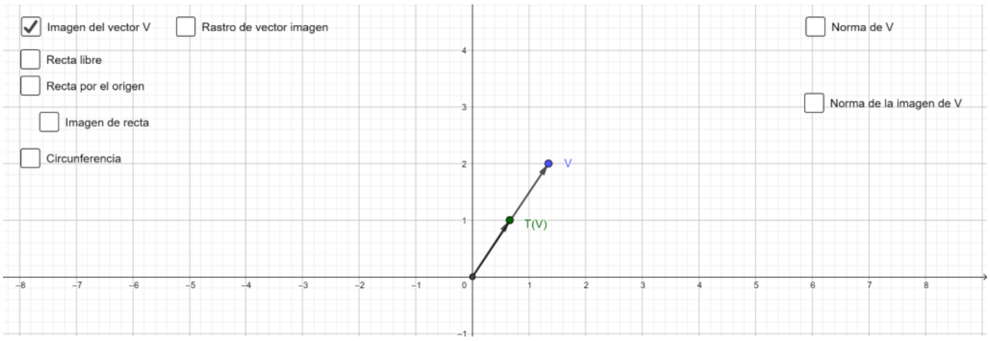

Figure 8. Collinear movable vector and its image in the first quadrant of the plane

Continuing the same way, the instructor concluded that the X-axis is also an invariant subspace. Finally, he calculated the equations of the straight lines representing the subspaces: $y = \frac{3}{2}x$ and $y = 0$ by observing the slopes.

From the perspective of MWS theory, Fernán's mathematical work started with the activation of the Semiotic-Instrumental plane. The instructor recognized the arrows anchored to the origin in the dynamic environment as elements of the vector space $\mathbf{R}^2$, initially represented in an algebraic register; this shows the activation of the semiotic genesis. He identified the arrow v as an element of the transformation's domain and the arrow $T(v)$ as an element of its image. The cognitive process of visualization allowed the instructor to understand the manipulation of the mobile vector not just as the dragging of an arrow, but as a way to explore the graphical behavior of the linear transformation through the observation of the images of different vectors of the domain.

The semiotic genesis of Fernán's mathematical work coexists with and is related to the instrumental genesis. The instrumental genesis was manifested when Fernán generated use schemes in the instrumentation process by rotating the mobile vector as a dragging technique. When he detected the collinearity of v and $T(v)$, he pondered his strategy to determine invariant subspaces, overcoming the discrepancy between his initial strategy of looking for a basis vector whose image is itself and the possibility of v and $T(v)$ being collinear but different vectors. During the exploration, another cognitive process of construction was identified when Fernán referred to the subspaces generated by v y $T(v)$, and he realized that they were the same.

The instructor also activated the Semiotic-Discursive plane in this part of his mathematical work. The semiotic genesis and the discursive genesis are related and underlie the “without seeing” visualization of the invariant subspace detected by Fernán. We call it “without seeing” because the subspace didn't appear on the screen of the environment. The instructor organized the information obtained from the dynamic environment to interpret and enrich his ideas. Reflection about the appropriate representations – collinear vectors – favored the elaboration of a discourse to explain the invariant subspaces in the given circumstances using the artifact GeoGebra without the need to visualize the subspaces in the dynamic environment. The conscious translation of the definition provided by Fernán from an algebraic representation into a geometric-dynamic representation was not instantaneous; nevertheless, the instructor could interpret and extend his ideas thanks to the three active geneses and the intervention of his previous theoretical referential.

From the perspective of APOS theory, expressing ideas about a concept using more than one representation, and transition between these representations requires a process conception. The instructor went from an algebraic representation of a variable vector of an abstract subspace to a graphic representation with a mobile arrow in the dynamic environment, showing evidence of a process conception of vector.

The dragging of the mobile vector implies an Action in the dynamic environment; however, we also note that the instructor was aware that he was manipulating a generic vector as an Object, not a particular given vector. Dragging the mobile arrow meant to him traversing one part of the domain vectors and applying the transformation (whose rule, he did not know) in order to obtain their images. This Action over the Object generic vector implies a Process conception of linear transformation. Fernán used it to explore the images of the vectors of a region in the domain, understanding that this exploration could be extended to the whole plane.

The instructor displayed an Object conception of subspace and Process conception of invariant subspace when he decided to apply the linear transformation to any subspace and from its image determine if it was invariant or not. To evaluate the linear transformation in any subspace, the instructor proposed to evaluate the transformation on some basis vectors. This strategy required the de-encapsulation of the generic one-dimensional subspace Object in its Process as being generated by a vector. Fernán initially proposed finding a vector whose image was itself; however, the use of the software induced the necessary adaptations in the invariant subspace Process, and the instructor came up with a new favorable strategy.

In Fernán’s case, the Process conception of invariant subspace arises from the coordination of the Processes of linear transformation and span (generated vector space) through set inclusion. Coordination was manifested when the instructor imagined any vector of the subspace as a linear combination of a vector v selected as a basis vector and then applied the transformation to each of the elements of the generated subspace, noting the relationship that the image set holds with respect to the original set. Since $T(v)$ y v are collinear, his Process conception of linear transformation with a focus on the $T(\lambda v) = \lambda\, T(v)$ property allowed Fernán to conclude that the subspace generated by v was invariant. As a particular characteristic, the instructor de-encapsulated mentally the Object span without manipulations in the dynamical geometry environment.

7.2.2. Relating eigenvectors and invariant subspaces

In traditional teaching of linear transformations and related concepts, the use of associated matrices is very common. In this context, Fernán explained that the eigenvalues of a linear transformation are the zeros of the characteristic polynomial and presented the algorithm to obtain it from an associated matrix $M_{f.}$ He referred to the eigenvectors as those x that satisfy the equality $M_f \times x = \lambda\, x$ where λ is a scalar. The instructor called the eigenvectors as "invariant vectors under the transformation", even though this is not appropriate mathematical terminology.

When we asked Fernán how he could identify the eigenvectors of the linear transformation in question using the dynamic environment, he said: "I have no clue, I had never thought about it in geometrical terms, give me two seconds. I have not thought about it geometrically". Recalling his idea about "invariant vectors", he explained that the span of an eigenvector is an invariant subspace.

F: If you have an eigenvector and take the span of that eigenvector, when you apply the transformation to it, it remains invariant. Why? Because the $T(v)$ of that vector is λ times that vector. So [the vector and its image] remain in the same direction… Not in the same direction, in the same subspace (he draws a line in the space), right? So, if each one of the elements of the basis fulfill that their image is λ times that vector, any vector of the span is going to fulfill the same.

To start thinking graphically, Fernán considered an eigenvector as the basis of a one-dimensional subspace. He argued the collinearity of the basis vector v and its image $T(v)$ by the definition of eigenvector and said that he could think about λv in geometrical terms as either a dilation or shrinkage in the same direction as v or in the opposite direction.

From the theoretical perspective of MWS, we observe that the mathematical work of Fernán belongs to the Semiotic-Discursive plane. To think about eigenvectors and eigenvalues in a graphical way, the instructor searched for relations between eigenvectors expressed through algebraic representations $T(v) = \lambda v,$ and the concept of invariant subspace recently interpreted in the dynamical environment. The semiotic genesis was manifested when he thought about the span of an eigenvector and the span of its image, intertwining algebraic and graphic representations.

Fernán considered the definition of eigenvector in algebraic terms as the vector that satisfies $T(v) = \lambda v$. After incorporating mental representations of graphical nature, he interpreted v and $T(v)$ as collinear directed segments. This change of representation displayed in the semiotic genesis allowed the instructor to link the concepts of eigenvector, span, and invariant subspace.

From an APOS perspective, we note that Fernán used his Process conception of invariant subspace to determine that the span of an eigenvector v is a subspace of the same kind. This Process conception involved the Processes of span and linear transformation. He followed his idea of generating a subspace with an eigenvector and examining its image after applying the linear transformation. Fernán's explanation shows that he used his Process conception of span as he worked with a generic eigenvector v to obtain $\langle v \rangle$, then applied his Process conception of linear transformation to the subspace $\langle v \rangle$ which he had constructed as an Object, and analyzed its image. The instructor used the property of preservation of a scalar multiple when applying his Process conception of linear transformation to the vectors of $\langle v \rangle$.

7.2.3. Eigenvalue detection in a dynamic environment

After his work with the dynamic environment and the spans of eigenvectors, Fernán claimed that eigenvectors constituted the invariant subspaces without providing any argumentation; we note that this claim does not necessarily hold for subspaces of dimension greater than or equal to 2. The instructor said that all the vectors that integrate the straight lines $y = \frac{3}{2}x$ and $y = 0$ are eigenvectors of the linear transformation presented in the dynamic environment; however, the mathematical convention does not admit zero vectors as an eigenvectors.

We asked Fernán how eigenvalues could be identified using the dynamic environment.

F: Aaah, no. Yes, yes. For example, like in the example (signals the vector (1, 0), see Figure 9), I have for $\lambda = -1$, which should be one of the roots of the characteristic polynomial, I have this eigenvector. Because it sends v to its negative… For $\lambda = 1$ you have this, look. (Drags the vector v to the position (–1,0) and observes its image). Ah no, it is also –1. If you have $\lambda = 2$ (drags the mobile vector to (2,0) and observes its image (–2, 0). Aaah, no. All of these are $\lambda = -1$. It doesn’t matter where I move, it will always stay at its negative.

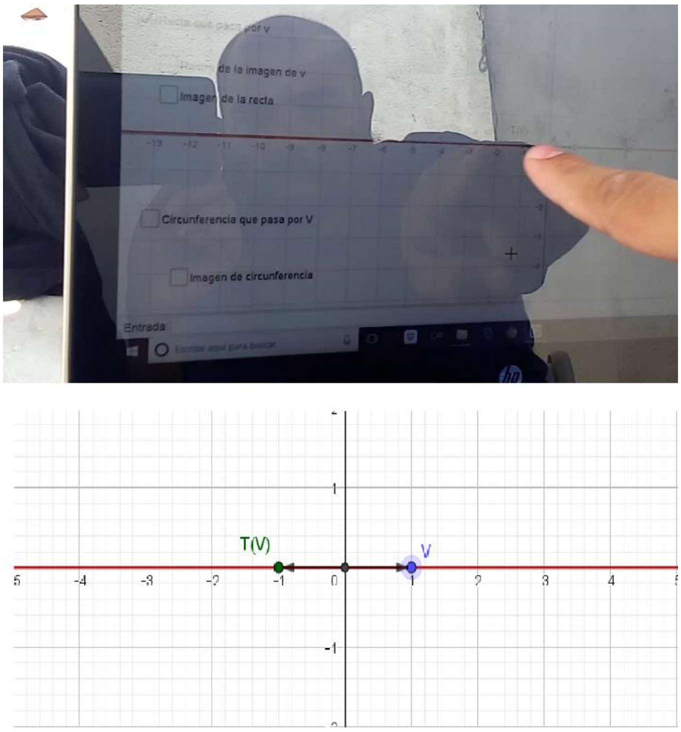

Figure 9. The instructor points to the vector (1,0) to show an eigenvector associated with $\lambda = -1$

For his answer Fernán directed his mathematical work to the Semiotic-Discursive plane; he recognized the scalar –1 as a possible eigenvalue by comparing the arrows representing $v = (1,0)$ and $T(v) = (-1, 0)$ in the dynamic environment. It might appear that the instructor considered the lengths of the arrows and their direction as graphical aspects of v and $T(v)$ associated to the eigenvalue concept; however, we also observe that Fernán considered different eigenvalues for different eigenvectors of the line $y = 0$. The relationships that he established between the concepts of linear transformation, vector, eigenvector, and eigenvalue of his theoretical referential were not yet solid.

Fernán carried and translated the definition of eigenvector – if v is an eigenvector, then $T(v) = \lambda v$ – from an algebraic register to a graphical register. He was thinking about v and $T(v)$ as collinear vectors but without reflecting carefully on the role of λ in the new register. He reasoned neither algebraically nor graphically about the constraint that all vectors of the invariant subspace, accepted as a set of eigenvectors, have as images multiples of themselves by the same scalar λ. The graphical aspects recently added to his theoretical referential – but not yet integrated – were not in line with the algebraic aspects of his established ideas about eigenvectors and eigenvalues. There was no semiotic genesis but instead attempts by the instructor to reinterpret the algebraic ideas of his theoretical referential in terms of the representations visible in the dynamic environment.

From the point of view of APOS theory, when the instructor identified the eigenvalue –1 by observing the vector $v = (1, 0)$ and its image $T(v) = (–1, 0)$, he displayed an Action conception of this concept stimulated by the perception of the graphic representations of the vector and its image. Fernán compared v and $T(v)$, assigning the eigenvalue –1 because the image of v is a collinear vector with the same length but opposite direction. However, the instructor revealed that he did not have a Process conception when trying to find in the same subspace a vector whose associated eigenvalue was 1 and then a vector with eigenvalue *2*. Actually, Fernán showed that his initial decision to associate the eigenvalue –1 to (1, 0) does not come from a Process conception, by suggesting the scalar 2 as the eigenvalue related to (2, 0). He considered the length of v without considering its relation to the length of $T(v)$.

8. Discussion

The study of one-dimensional invariant subspaces of a vector space under a linear transformation provides an alternative way for introducing the concepts of eigenvector and eigenvalue, as opposed to defining an eigenvalue as a root of a characteristic polynomial and only afterwards defining an eigenvector in terms of an eigenvalue. This approach helps establish connections between different notions of linear algebra such as basis, span, linear transformation, vector space, subspace, eigenvector and eigenvalue; it also becomes much more significant in terms of the theoretical structure, to study the effect of a linear transformation on the whole plane through the invariant subspaces. However, as observed in the interviews that we conducted, these connections are not always evident, even for instructors of linear algebra. In fact, the great majority of undergraduate students are not familiarized with the notion of invariant subspace through an introductory linear algebra course.

Dealing with various representations of the same concept helps integrating different aspects of a notion and hence with the understanding of it. Dynamic representations of linear algebra objects can aid in facilitating mental construction of concepts and coordination of several Processes. Both instructors that we interviewed showed a certain resistance to the use of dynamical geometry software; this might be partially due to their being accustomed to working within the algebraic register and relying on traditional methods of teaching. However the influence of this environment on their thinking was obvious, even though less evident in the case of Daniel because of the order of the proposed activities.

The mathematical work of the first instructor Daniel in the Semiotic-Discursive plane became primordial for validating the line as a subspace. The semiotic genesis, nourished with the instructor's mental structures associated with some existing concepts in his theoretical referential, allowed Daniel to verify that the straight lines passing through the origin are subspaces. The connections between mental structures were achieved thanks to the semiotic genesis; through the manipulation of signs, the instructor validated the straight line as representing a subspace of R^3. The treatments provided coherence to the instructor's subspace Schema. In other circumstances, possibly the mental structures associated to the Processes of line through the origin, span, and subspace, expressed algebraically, would have remained isolated without the sign treatments that allowed the visualization of the line as a subspace.

Now, the subspace represented by the equation $y = 0$ is not invariant in the same way that the subspace defined by $y = \frac{3}{2}x$ is invariant. In both cases, the image of the line is the same line. However, the effect of the transformation on the elements of the first subspace is not the same as in the second. The characteristics of two one-dimensional *T*-invariant subspaces might be perceived as identical globally if each subspace is an image of itself. However, within the subspace, the function's behavior might be different. Realizing this requires the de-encapsulation of an Object to the Process from which it came from.

Considering the effects of the transformation interior to the invariant subspace might involve some difficulties. For example someone might think that the image of a 2-dimensional *T*-invariant subspace in R^3 must be the same as itself. However, this individual would not be considering the following subspaces as possible images: a line contained in the same plane and passing through the origin, the origin, or the same plane but with all its vectors in a different position. This might be an obstacle for the individual in distinguishing whether an invariant subspace is an eigenspace. We think of these considerations as a valuable opportunity to expand the schema of linear transformation by designing a mathematical working space that favors the interiorization of graphical Actions related to the invariant subspace concept.

For the most part, Daniel’s mathematical work sidestepped the analysis of the graphical properties of invariant subspaces to resort to the computational aspects associated with the eigenvector and eigenvalue concepts in the algebraic register. The fragility of his graphical constructions related to the concepts eigenvector, eigenvalue and invariant subspace led the instructor to resort to a conjecture to explain the eigenvectors in the dynamic environment: to determine the eigenvectors, it is enough to find the invariant subspaces. It seems that when reasoning about graphical aspects, Daniel separated the notions of eigenvector and eigenvalue, since, although he perceived the infinity of eigenvectors, he was not interested in the characteristics of $T(v)$ for any v.

Daniel did not reflect about the image vector of an eigenvector being linked to its associated eigenvalue. If the instructor had explored the images of invariant subspaces in the software, by means of the execution of graphic Actions he could have thought about the relationships between an invariant one-dimensional subspace and eigenspace, but this did not happen. We think that the design of a mathematical working space that promotes the activation of the semiotic and instrumental geneses can facilitate the interiorization of graphic Actions related to the eigenvector and eigenvalue concepts, where the use of the scalar multiplication property of linear transformation Process is indispensable. Figure 10 shows some of the circulations in Daniel's mathematical work.

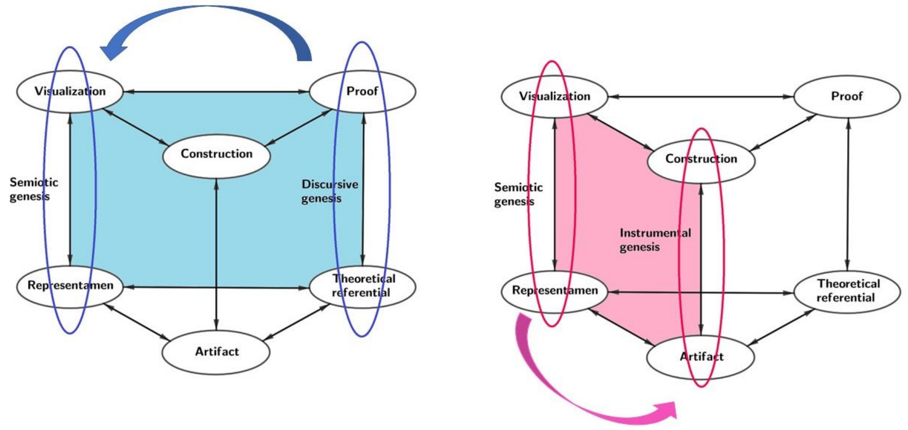

Figure 10. Circulations in the mathematical work of Daniel when determining invariant subspaces of *T* employing the dynamic environment

In Fernán's case, the instrumental and semiotic geneses in his mathematical work with the software favored the interiorization of his invariant subspace Actions, which consist of evaluating a linear transformation on a given subspace and deciding whether its image is contained or not in that subspace. Posing this problem in an algebraic register would require knowing the rule of the linear transformation. This might imply reducing the concept to specific or prototypical cases due to operational difficulties with complex expressions. We observed that the instrumental and semiotic geneses favored the interiorization of the Action structure related to invariant subspace, promoting reflection on its execution, which could hardly be carried out repeatedly in a paper and pencil environment. We also hypothesize that the construction of schemes related to the use of dynamical geometry software can motivate reflection on the existence of invariant subspaces whose image is the zero vector, absent from the instructor's thoughts during the interview.

Another example of the use of mental mechanisms through the activation of geneses occurred when Fernán explained a relationship between eigenvectors and invariant subspaces. To demonstrate that the former constitute bases for the latter, Fernán activated a semiotic genesis to transit between representations. This semiotic genesis favored for him the coordination of the mental processes related to different concepts of Linear Algebra. We hypothesize that designing a mathematical workspace that allows the activation of the three geneses having semiotic and instrumental geneses as core, would contribute to constructing the eigenvector Process.

Additionally, we think that the activity in the Semiotic-Instrumental plane related to the direct manipulation of the dynamic environment can also promote the mechanism of interiorization of the graphical Actions of eigenvector by exploring the vectors of an invariant subspace. The Process resulting from this interiorization would allow the construction of the eigenvalue Process by means of the reversal mechanism. This mechanism can also germinate within the mathematical work developed in the Semiotic-Discursive plane since it implies reflection on the Actions that must be performed on the scalar to obtain a vector that grants it the character of eigenvalue. Figure 11 shows some of the circulations in Fernán's mathematical work.

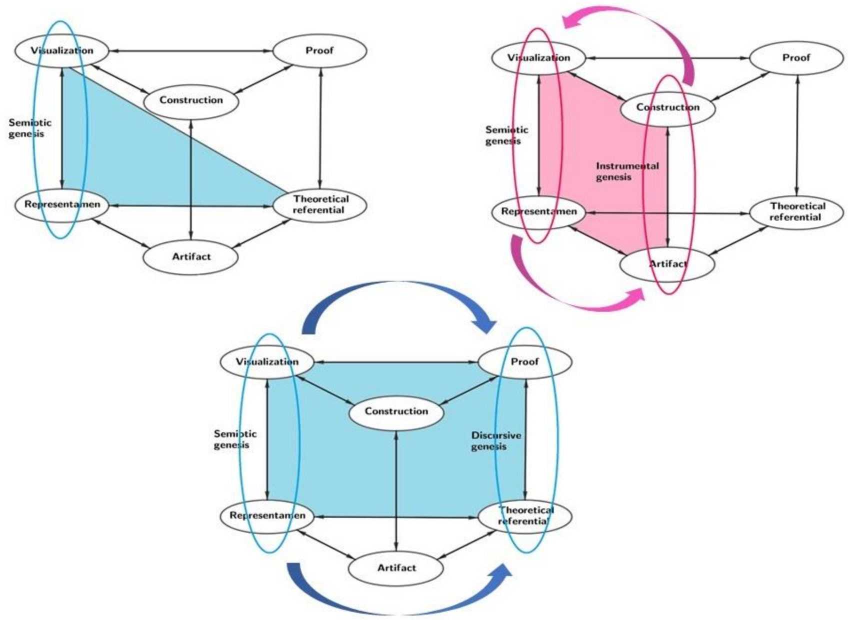

Figure 11. Circulations in the mathematical work of Fernán when detecting invariant subspaces

It should be clarified that circulation through the three planes involved in the personal mathematical working space of an individual, as well as the activation of the three geneses, do not guarantee the construction of mental structures associated with the concepts of eigenvector and eigenvalue. During the interview, Fernán evidenced circulations in the three different planes and activation of the three geneses and did not demonstrate a Process conception of either eigenvector or eigenvalue, at least graphically. It should also be noted that moving between representations, in the case of semiotic genesis, does not ensure a structural and conceptual understanding either. Conversion of registers implicates Processes; otherwise, there would only be translations of fixed data without access to the structural richness of the concepts.

Our exploration through interviews with linear algebra instructors showed a promising approach for introducing and developing eigenvector and eigenvalue concepts by means of their interaction with the notion of invariant subspace. We recommend incorporating this approach in undergraduate courses through the use of dynamic geometry and careful design of mathematical situations.

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