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EARLY CHILDHOOD MATHEMATICS EDUCATION RESEARCH: OVERVIEW OF LATEST DEVELOPMENTS AND LOOKING AHEAD

Abstract. In the present study, we provide an overview of the research in the field of early childhood mathematics education and identify the latest advances, new perspectives and gaps in the literature between 2012 and 2022. On the basis of our review of the international research literature published during this time span, in this paper we focus on five major themes of contribution: young children's number sense abilities and development, geometry education in early childhood, children's competencies in other content domains, teaching and learning mathematics in early grades with technology-integrated activities and early childhood teachers' knowledge, education and affective issues in mathematics. For each theme relevant research is discussed and directions for future research are provided.

Key words. Literature review, surveys, early years, number sense, geometry, technology, early childhood teachers, young children

Résumé. Recherche sur l'enseignement des mathématiques à la petite enfance : aperçu des derniers développements et perspectives. Dans cette étude, nous proposons une vue d'ensemble de la recherche dans l'enseignement des mathématiques de la petite enfance et nous identifions les dernières avancées, les nouvelles perspectives et les lacunes dans la littérature de 2012 à 2022. À partir de notre revue de la littérature internationale publiée pendant cette période, nous nous penchons sur cinq grands thèmes de contribution : les capacités et le développement du sens du nombre chez les jeunes enfants, l'enseignement de la géométrie dans la petite enfance, les compétences des enfants sur d'autres contenus mathématiques, l'enseignement et l'apprentissage des mathématiques dans les premières années grâce à des activités intégrant la technologie, ainsi que les connaissances des enseignants de la petite enfance et les enjeux éducatifs et affectifs en mathématiques. Pour chaque thème, nous discutons des recherches pertinentes et proposons des orientations pour la recherche future.

Mots-clés. Revue de la littérature, enquêtes, premières années, sens du nombre, géométrie, technologie, enseignants de la petite enfance, jeunes enfants

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1. Introduction

Young children's early mathematical knowledge and skills affect their later learning and success in mathematics (Dunkan et al., 2007; Watts et al., 2014). This well-documented finding, as well as the strong emphasis given on early childhood education in many countries (Kagan & Roth, 2017; Taguma et al., 2012; UNESCO,

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2015), have stimulated an internationally growing interest in early childhood mathematics education (ECME) research in the past few years and have highlighted the need for high-quality mathematics education in early childhood (Elia et al., 2021).

ECME is a broad and rich field of research and practice in the discipline of mathematics education which focuses on offering young children opportunities and experiences to learn mathematics and develop mathematical abilities and concepts through the provision of motivating activities and learning environments (Van den Heuvel-Panhuizen & Elia, 2014), which are organized by caretakers, teachers and other professionals (Björklund et al., 2020). The focus of recent research on early childhood mathematics education is on children from birth until their entrance to formal education in the first grade (Björklund et al., 2020). Taking a comprehensive perspective, in the present study[1] we aim to provide an overview of the research in the field of early childhood mathematics education and potentially identify the latest developments, new perspectives and gaps in the literature between 2012 and 2022 and also discuss opportunities for future research. For the purpose of this study, we focus on research which involves children from birth to the first schooling year, that is, 7 years of age. Children up to this age have not yet started formal education or are at the beginning of formal schooling in many countries. This age range of children is in accordance with the focus of the great majority of studies discussed in the ICME Topic Study Group and publications on ECME in the past few years, which involve mainly research on children’s mathematical development in the years until formal education (Elia et al., 2018; Elia et al., 2021; Van den Heuvel-Panhuizen & Kullberg, 2021).

In the past few years, there has been a small number of review papers related to ECME research, focusing on specific aspects of the field, including for example early years teachers (Linder & Simpson, 2018), children up to four years of age (MacDonald & Murphy, 2021) and the Australasian context (MacDonald et al., 2016). To the best of our knowledge, there has not yet been an overall survey of the international literature on ECME. This study is an outline of the state-of-the-art of the major advances in the field and could serve as the basis for initiating further systematic reviews in more specific noteworthy topics in the learning and teaching of early years mathematics and also for discussing future directions for research. It should be noted that the present study focuses primarily on research undertaken in the context of the English-speaking scientific community, which covers a large part

[1] This paper is based on a survey addressing the latest developments in ECME that was initiated in the context of the 14th International Congress on Mathematics Education (ICME-14) and a part of the findings of the survey were presented at the conference on July 18, 2021.

of the work in the field and has a high level of accessibility. Nevertheless, we acknowledge that there is important and relevant research work beyond this geographical and cultural context, including, for example, literature in the French language. This limitation of the study could be addressed in future reviews of studies carried out by non-English-speaking researchers which may complement, broaden and enrich the findings of the present study.

2. Method

Our overview was completed in four steps: identification of relevant themes in ECME research, search of literature per theme, selection of publications per themes and analysis of the selected publications per theme (see Figure 1).

A first step of this study was to identify, as a team, the possible themes of contributions to the field of ECME recent research based on our shared expertise. As a result, six major themes were articulated, as follows: three content-oriented themes, namely, young children's number sense abilities and development, geometry education in early childhood, children's competencies in other content domains; a theme on teaching and learning mathematics in early grades with technology-integrated activities; a theme on early childhood teachers' knowledge, education and affective issues in mathematics; and a cognition-oriented theme on cognitive skills associated with mathematics learning and special education. In the present paper, we will focus on the former five themes.

The next step was to search for relevant research literature on each theme published from 2012 through 2022. As noted previously, we searched only for publications written in English. For each theme, we conducted a database search of Scopus, Web of Science, Google Scholar, ERIC with a focus on journal papers in three fields of study: Mathematics Education, Early Childhood Education and Psychology/Cognitive Science, using relevant keywords. Across all the themes, the keyword "mathematics" was used, which was combined with one of the following terms "early childhood", "preschool", "kindergarten", "young children", "early years". For each theme, these keywords were combined with theme-specific terms, e.g., "technology" or "digital tools" etc. for the technology theme. Additionally, we searched for and recommended relevant chapters in prominent research books and monographs in Mathematics Education and child development, including the PME Handbook, POEM and the ICME-13 monographs.

As a third step, for each theme we eliminated double records, and after reviewing the title, abstract and keywords of the remaining papers we eliminated further papers based on specific exclusion criteria, that is, studies which did not focus on mathematics and studies with children above seven years of age. The review per theme included empirical research studies, reviews and theoretical papers.

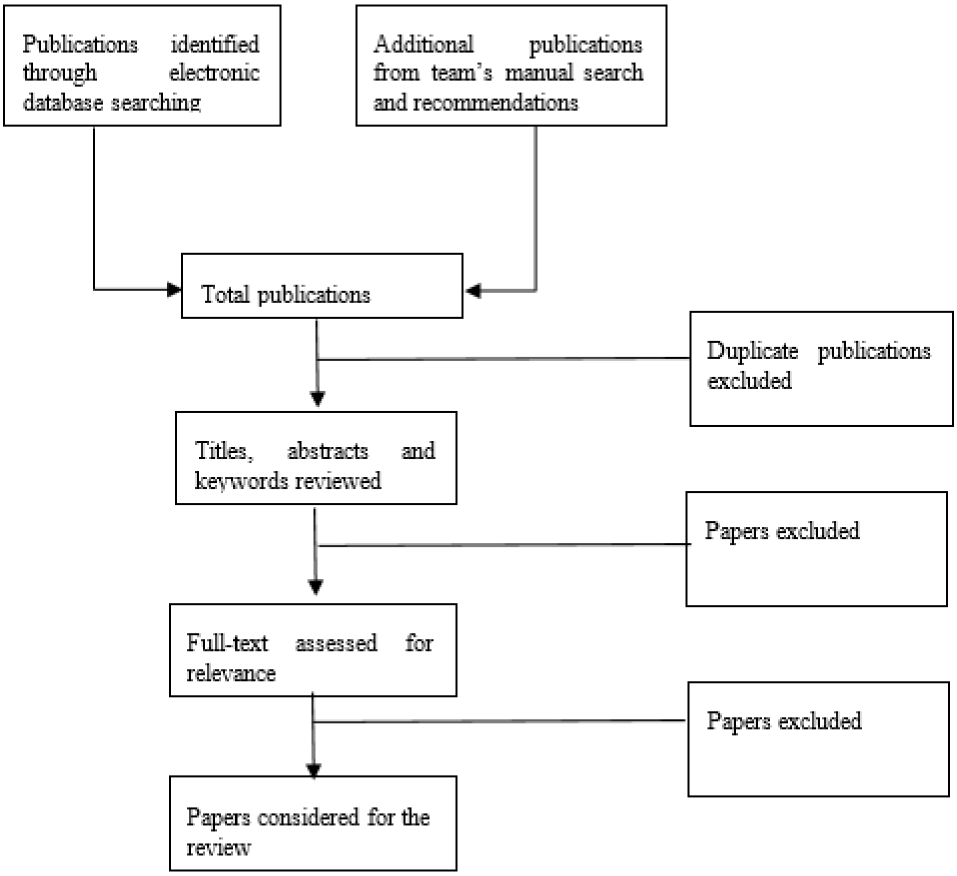

Figure 1. Search process of publications per theme

We then read every study, eliminated further papers based on the same criteria as noted earlier, and analysed the remaining studies which were identified as relevant for each theme to produce annotated bibliography with summaries, focusing on the purpose and the key findings of each study. Next, we used an inductive coding approach (Thomas, 2006) to analyse the summaries in order to identify categories for the review of each theme. Following this, an in-depth reading of each paper per theme was undertaken in order to identify major findings for each category. This led to a qualitative synthesis of the pertinent findings of the research for each theme category. The categories that emerged for each theme and the synthesis of the key findings of the relevant literature are included in the results and discussion section. Because of the number of themes in the ECME research addressed in this study (n=5), the wide scope of each theme and the vast amount of relevant research within each theme, it is not feasible to include all the studies that were found and analysed in this paper. For every theme, the paper includes a selected sample of the studies that have been reviewed, which, to our knowledge, point to new developments and perspectives compared to existing research.

3. Results and discussion

The results reported here are based on a selection of the studies that were analyzed for each theme category, as mentioned above. Tables 1-3 detail the emerging categories, the quantity and the sample of publications that are included in the present paper for each category within the content-oriented themes (see Table 1), the technology theme (see Table 2) and the teachers' theme (see Table 3), respectively. The total quantity of papers that were included in the annotated bibliography and thus in the analysis per theme are also provided.

Table 1. Categories per content-oriented theme from reviewed papers and selected publications

Category	Number of selected publi-cations included in the paper	Sample of publications
Theme: Young children's number sense abilities and development (n=128)[a]		
Young children's numeracy abilities	10	Benz (2014); Dehaene-Lambertz & Spelke (2015); de Hevia et al. (2014); Hannagan et al. (2017); Reikerås et al. (2012); Robertson et al. (2012); Schöner & Benz (2018); Sella et al. (2016); Sella et al. (2017); ; Spaull et al. (2022)
Enhancing diverse numeracy abilities of young children	24	Asakawa et al. (2019); Bay-Williams & Kling (2014); Bicknell et al. (2016); Casey et al. (2018); Clements et al. (2020); Cui et al. (2017); Fuchs et al. (2013); Gaidoschik (2012); Harvey & Miller (2017); Hermawan (2021); Holmes & Dowker (2013); İvrendi (2016); Jordan et al. (2012); Lüken & Kampmann (2018); Magnusson & Pramling (2018); Polotskaia & Savard (2018); Ramani & Siegler (2011) ; Sayers et al. (2016); Segers et al. (2015); Spaull et al. (2022); van Marle et al. (2014) ; White & Szucs (2012); Widodo & Yusuf (2022); Xu et al. (2013)
Contribution of Spontaneous Focussing on Numerosity (SFON) to Numerical Abilities	3	Batchelor et al. (2015); Rathé et al. (2018); Torbeyns et al. (2018)
Insight on the role of Approximate Number System in numerical knowledge	11	Bonny & Lourenco (2013); Chu et al. (2015); Peng et al. (2017); Sasanguie et al. (2014); Sullivan & Barner (2014); Van Herwegen et al. (2017); van Marle et al. (2018); Wong et al. (2016)
n[b]	44	

Theme: Geometry education (n=70)		
Spatial reasoning and early geometry	5	Dindyal (2015); Hallowell et al. (2015); Kaur (2015); Soury-Lavergne & Maschietto (2015); Woolcott et al. (2022)
Shape knowledge and understandings	6	Dağlı & Halat (2016); Halat & Daği (2016); Hallowell et al. (2015); Kalénine et al. (2013); Resnick et al. (2016); Swoboda & Vighi (2016)
Dynamic, embodied and semiotic approaches in geometrical thinking and learning	13	Breive (2022); Bussi & Baccaglini-Frank (2015); Calero et al. (2019); Dindyal (2015); Elia et al. (2014); Gejard & Melander (2018); Kaur (2015); Kaur (2020); Moss et al. (2015); Ng & Sinclair (2015); Thom (2018); Thom & McGarvey (2015)
Enhancing and assessing geometry learning	13	Bäckman (2016); Casey et al. (2014); Cheng & Mix (2014); Fisher et al. (2013); Hawes et al. (2015); McGuire et al. (2021); Nakawa (2020); Nurnberger-Haag (2017); Thom (2018); Thom & McGarvey (2015); Van den Heuvel Panhuizen et al. (2015); Verdine et al. (2014); Verdine et al. (2017)
Individual differences associated with learning in geometry	5	Jirout & Newcombe (2015); Milburna et al. (2019); Mushin et al. (2013); Rittle-Johnson et al. (2017); Verdine et al. (2014)
n^b	39	
Theme: Other content domains (n=56)		
Patterns	14	Bäckman (2016); Björklund & Pramling (2014); Collins & Laski (2015); Hunter & Miller (2022); Kidd et al. (2014); Lüken & Kampmann (2018); Miller (2019); Mulligan & Mitchelmore (2013); Mulligan et al. (2020) ; Rittle-Johnson et al. (2015); Swoboda & Vighi (2016); Tsamir et al. (2017); Venkat et al. (2018); Wijns et al. (2019)
Measurement	4	Clements et al. (2018); Kotsopoulos et al. (2013); Sarama et al. (2021); Szilágy et al. (2013)
Spatial reasoning	11	Casey et al. (2014); Cheng & Mix (2014); Gold et al. (2021); Kotsopoulos et al. (2021); Laski et al. (2013); Möhring et al. (2015); Resnick (2020); Van den Heuvel-Panhuizen et al. (2015); Verdine et al. (2017); Woolcott et al. (2022); Zhang & Lin (2017)

Thinking, reasoning and data modeling	7	Bakker et al. (2014); Blanton et al. (2015); English & Crevensten (2013); Kieran et al. (2016); Lenz (2022); Obersteiner et al. (2015); Supply et al. (2021)
n[b]	35	

[a] Total number of papers analyzed

[b] Number of papers included in the paper (due to dual coding this number may be smaller than the sum of papers from all categories per theme)

3.1. Young children's number sense abilities and development

Number sense development is globally recognized as the fundamental foundational knowledge for children's mathematical growth. To this end, literature argues for children's stimulation of numerosity at an early toddler stage for the children's future benefit. The complexities brought forward by diverse backgrounds of children as well as diverse provisions of stimulation enrich strategies and seek more conceptualisation. Children's diverse numerical abilities reflect children's varied experiences from home and their immediate environment (Ramani & Siegler, 2011). These abilities are foundational blocks for children's development of numerical fluency, and anecdotal evidence show a correlation between low performance as causative to limited experiences prior kindergarten and inability to catch up with peers (Aunio et al., 2015).

3.1.1 Young children's numeracy abilities

The importance of innate abilities and non-symbolic abilities are highlighted as important concepts to be included in defining number sense of young children (Dehaene-Lambertz & Spelke, 2015). Literature indicates that these innate abilities of young children are observable from their infancy (Dehaene-Lambertz & Spelke, 2015; Hannagan et al., 2017). Newborns of 7 to 94 hours demonstrated that they could connect increases in number of objects with increases in length and in time (de Hevia et al., 2014). Robertson et al. (2012) discovered that 24 months babies were able to match objects with the defining number. Norwegian toddlers demonstrated competencies using number words, however, reciting was found to be at lower competencies than suggested previous literature (Reikerås et al., 2012).

Sella et al. (2016, 2017) explored numerosity and spatial mapping to three groups, preschool children, 4-year-olds, 1st Grade and 3rd Grade, to discover that spatial mapping favoured high numerical abilities in all groups studied. A comparison of numerical abilities between Finnish and Iranian children from 5 to 6 years old revealed that relational and counting tasks scores favour Finnish children compared to Iran. Schöner and Benz (2018) determined that children build structures in the collection of objects but cannot explain their approaches and resort to counting as a strategy, whereas Benz (2014) revealed that 4- to 6-year-olds were able to explain structures in quantities and why they used to compose or decompose. On the other hand, low socio-economic variable seems to bring a different tone to numerical abilities of young children. A South African study of numerical abilities of first grade students from no fee schools of Limpopo and Eastern Cape Province indicated that the majority operated at a low level of counting while only 30% understood that numbers can be decomposed into smaller units (Spaull et al., 2022).

3.1.2 Enhancing diverse numeracy abilities of young children

The context and background of students contribute significantly to their numerical abilities as reported by Spaull et al. (2022). Hence, it is crucial to explore strategies to enhance and develop students' numerical abilities in these different contexts. Diverse strategies that are revealed to enhance numerical abilities with positive gains on numeracy development of children are linear board games (Ramani & Siegler, 2011), numerical acuity (van Marle et al., 2014) and inhibitory control (Harvey & Miller, 2017), conceptual subitising (Sayers et al., 2016), fine motor skills (Asakawa et al., 2019), number knowledge tutoring (Fuchs et al., 2013), story problems (Jordan et al., 2012), catch up numeracy (Holmes & Dowker, 2013), differentiated approach in using games (Bay-Williams & Kling, 2014), and music-based learning (Hermawan, 2021). A recent research study by Clements et al. (2020) showed that the learning trajectory approach contributed to kindergartners' learning of addition and subtraction to a greater extent compared to a teach-to-target approach. Moreover, significant benefits are observed in play-based approach for numeracy development at home, allowing children to learn through play and give opportunity to adults to pose challenging questions and listen to children illustrating their action and adding meaning to them (Magnusson & Pramling, 2018; Widodo & Yusuf, 2022).

Some concept development also plays a significant role in the development of numerosity. Mediation of patterns and structure was found to influence children's numerical fluency positively especially those who were low performers (Lüken & Kampmann, 2018). Problem solving boosted four basic operations with an effect size of 0.60 (Bicknell et al., 2016). Polotskaia and Savard (2018) used Relational Paradigm in facilitating problem solving and this led to improved problem-solving skills and enabling students to solve problems demanding rational thinking. A longitudinal study favoured students who began to use derived fact strategies during the mid-year than those using counting strategies (Gaidoschik, 2012). White and Szucs (2012) promoted modelling methods to increase understanding and developing mental representation through estimation tactics. Then Xu et al. (2013) brought forth age as a variable that allows complex estimation skills. Their findings affirmed the accuracy and linearity of number estimates. Rapid automized naming (RAN) has been proven to predict reading skills and later literature indicates that it also predicts mathematical skills. Cui et al. (2017) investigated the link between RAN and arithmetic fluency of selected Chinese children. The association proven was between addition and subtraction fluency and RAN supporting that RAN predicted mathematics.

Mothers of 36 months' babies were claimed to have an influence on attainment of higher mathematics knowledge in the early years of schooling through stimulus to label magnitudes of different groups of objects (Casey et al., 2018). A study supported the latter study as the home numeracy environment (including home activities, such as counting objects) was found to predict kindergartners' early numeracy competence (Segers et al., 2015). Young children's self-regulation and number sense attested to be strong predictors of mathematics performance at grades 5 and 6 (İvrendi, 2016).

3.1.3 Contribution of Spontaneous Focussing on Numerosity (SFON) to Numerical Abilities

Spontaneous focussing on numerosity (SFON) by children comes naturally and carries possibilities for numeracy development in children. Rathé et al. (2018) and Batchelor et al. (2015) explored the association between 4- to 5-year-old children SFON and their number-related verbal words during their natural activities. The analysis showed varied differences within and between children and these indicated that SFON had no association with the frequency of number-related utterances, whereas Batchelor discovered that the symbolic numerical ability was associated with SFON. The latter findings were regular to previous work while Rathé et al. (2018) was contradictory. Hence, they used two experimental tasks to assess the association between SFON of 4- to 6-year-olds and their number utterances. The findings supported the relation between children's verbal SFON and number. A similar study was conducted in Ecuador by Torbeyns et al. (2018) and the numerical abilities of children were directly associated with their SFON, providing evidence for the universal character of the link between SFON and early quantitative competence.

3.1.4 Insight on the role of Approximate Number System in numerical knowledge

Approximate number system (ANS) is an internalised process that logically allows one to be able to make sophisticated judgments in numbers, measurement etc. This system is associated with early arithmetic through numerical knowledge (Peng et al., 2017; van Marle et al., 2018). Sasanguie et al. (2014) affirmed previous findings that there is no relation between accuracy of children and non-symbolic number. Efficiency in number mapping is associated with higher ANS insight, which then enhances understanding of number symbols and arithmetic skills (Wong et al., 2016). Sullivan and Barner (2014) argued that there is no relation between counting and estimation ability in young children. Van Marle et al. (2018) assessed if measures of the ANS, object tracking system (OTS) or both are associated with the development of cardinal knowledge and discovered that children construct their understanding and rely on the ANS which goes with verbal counting structure. Accurate number illustrations of preschool children showed more advanced numerical ability (Bonny & Lourenco, 2013). According to Chu et al. (2015), ANS is introductory to mathematical development though cardinality is a strong predictor of mathematical achievement at the end of the first year of schooling. Van Herwegen et al. (2017) examined how to enhance young children's ANS abilities. Children who attended an intervention named PLUS were found to improve in ANS after training.

3.1.5 Concluding remarks

This review extends our understanding of numerosity/number sense of young children's innate abilities to strategies that assist in developing these abilities for successful numerosity experiences. This review also brings forth diverse interventions that are contextual and contribute in diversifying stimulation environments. There is a need for literature that demonstrates the link between the intuitive innate abilities and development of conceptual numerical abilities. This could contribute to improving early mathematics education in order to support and enhance young children's learning and development based on their possibilities even from infancy. Although a lot of correlation studies have been conducted to test some of the young students' abilities and formal mathematics, there are gaps in literature on how to develop such abilities when found limited. For example, Spontaneous Focussing on Numerosity (SFON) of 4- to 6-year-old correlation studies indicate its association with number-related verbal words, symbolic numerical ability across all studies. However, literature does not indicate how to develop this ability prior to 4 years of age. Literature needs to make a distinction, if any, or indicate if subitizing is within SFON or vice versa, or SFON is a stand-alone ability. Another example is the approximate number system (ANS) ability that gives promise to estimation and literature proves it to be a strong predictor of mathematics achievement through cardinality (e.g., Chu et al., 2015). A concept that needs nurturing, so far one study from this literature shares a successful intervention PLUS that increases ANS (Van Herwegen et al., 2017).

Observations indicate a need to understand or unpack children's language as some studies asserted that children were able to articulate their strategies and reasons behind their selection (Benz, 2014), while others reported that children were limited to counting in describing their strategies (Schöner & Benz, 2018). These contradictory findings are a clear indication for more studies looking at the same phenomena, such as students' ability to explain structures they build. Furthermore, there is too little literature on transitioning from informal numerosity (e.g., SFON) to formal numerosity and how mediation should be structured to achieve the transitioning.

Number sense development through technology is an important additional issue which has been studied in recent research and has been reviewed in the present study. The review of the literature on this issue will be discussed within the theme of the role of technology in ECME in section 3.4.

3.2. Geometry education in early childhood

Based on our review of research literature on geometry education in the early years, we identified five different categories/threads on which recent research is focused. Relevant research findings on these threads are categorized into the corresponding sections that follow.

3.2.1 Spatial reasoning and early geometry

Geometry is a mathematical content domain in which learning encompasses spatial reasoning to a great extent. In this paper, spatial reasoning, which has been found to be a major predictor of later academic achievement at school, is considered as the ability to identify and use the spatial attributes of objects and the spatial relations between objects (Bruce et al., 2017; Woolcott et al., 2022).

Although there is extensive research which provides evidence for the positive relationship between spatial reasoning and early mathematical competences and learning in general (see Section 3.3.3 within the theme of other content domains in ECME), the complex relationship (Dindyal, 2015) between spatial reasoning and geometrical knowledge and understanding of young children in particular has been scarcely researched. In a recent special issue of ZDM in geometry at the primary school, the findings of a number of studies have revealed different aspects of this relationship. Particularly, in their study with 7-year-old students, Soury-Lavergne and Maschietto (2015) found that spatial knowledge is the basis for building geometrical knowledge and understandings in problem solving situations. Kaur (2015) has stressed the need to introduce more systematically the drawing of straight lines to young children (7-8 years of age), as this could enhance their reasoning about 2D or 3D shapes and specifically dimensional deconstruction of shapes (see also Duval, 2005). In another study, Hallowell et al. (2015) investigated the mereological, optics, and spatial operations of 6-7-year-old children by plane and solid shapes. They concluded that practicing these operations would support the development of children's visualization abilities when working with geometrical figures.

3.2.2 *Shape knowledge and understandings*

Shape knowledge and understanding is an important aspect of geometry education. Swoboda and Vighi (2016) argued that the research on the understanding of geometrical concepts has focused on two main issues: the understanding of geometric figures and the functioning of these figures in space. They, also, claim that the problem of understanding geometric figures was indicated by Van Hiele's theory which is still worth considering. Particularly, research on secondary school pupils' conceptions of geometrical objects and relationships has shown that this knowledge is not well established. One of the reasons is very poor recognition of the way geometric knowledge develops in the early educational stages.

Recent studies which focused on investigating the understanding of specific shapes by preschool children, including triangles (Dağlı & Halat, 2016; Kalénine et al., 2013), rectangles (Kalénine et al., 2013) or squares (Halat & Daǧli, 2016), reaffirmed the findings of earlier research which suggested that children identified, classified and drew shapes based on the comparison with visual prototypes (e.g., with specific orientation and ratio characteristics) and showed a higher rate of failure when dealing with non-prototypical shapes. In line with these findings, Resnick et al. (2016) provided evidence to children's exposure to a limited number of shape categories and very few non-typical variants within those categories. Less attention is devoted to studying children's competences with 3D shapes. In their study on first graders' reasoning about solid and plane shapes, Hallowell et al. (2015) showed that children encountered difficulties in reconstructing a 3D geometrical object and using its regularities when needed.

3.2.3 *Dynamic, embodied and semiotic approaches in geometrical thinking and learning*

The introduction of geometry as a dynamic, spatial and imaginative subject rather than as a subject emphasizing shape recognition and classification in the early years is suggested by a number of researchers (e.g., Moss et al., 2015). In line with this perspective, in a review of literature in Geometry Education, Sinclair, Bartolini Bussi et al. (2016) identified the use and role of diagrams and gestures and the advances in the understanding of the role of digital technologies (including dynamic environments) as major threads of contributions.

Focusing on literature in early geometry, there is growing research which investigates the representation of geometrical ideas (Dindyal, 2015), the different communicative modes that children use to describe shapes (i.e., talk, gesture, diagrams and material environment) (Gejard & Melander, 2018) and semiotic mediation (Bussi & Baccaglini-Frank, 2015) in the teaching and learning of geometry.

A common rationale of this research is that embodiment and multimodality are key aspects of early geometry learning, as speech alone might not be sufficient for expressing and organizing geometric concepts and therefore children pursue multiple paths to overcome its limitations (Calero et al., 2019). Particularly, a number of studies provide evidence for the crucial role of the bodily actions and gestures in various aspects of geometry learning, including children's spatial-geometric reasoning and conceptions (Thom, 2018), mathematical generalisation and abstraction of symmetry (Breive, 2022), making sense of shape and space concepts and communicating geometrical and spatial relationships (Elia et al., 2014). Furthermore, Calero et al. (2019) suggested that children's gestures and behavioral choices may reflect implicit knowledge and serve as a foundation for the development of geometric reasoning.

A number of studies have investigated the complex interplay between embodiment and other modalities in the process of geometrical meaning making. Most of these studies (Elia, 2018; Gejard & Melander, 2018) provided evidence for the synergy between talk, gesture, and material environment, where talk and gesture mutually elaborate upon each other as part of collaborative meaning-making practices.

A smaller number of studies investigated young children's geometrical thinking through their drawings. Particularly, Thom and McGarvey (2015) examined the ways that children come to draw in geometric contexts and they concluded that children's drawings in geometric contexts should not be seen simply as an outcome (after-the-event artifact), but as visual and kinetic geometric tools to present, conceptualize, and solve problems which contribute to their geometrical understandings.

The role of dynamic learning environments has been the focus of a number of studies in early geometry. Specifically, evidence has been provided for the potential of dynamic geometry environments to support children's developing discourse, understanding, and reasoning about, the properties and behaviours of shapes, particularly triangles (Kaur, 2015). Furthermore, the interplay between the use of dynamic learning environments and embodied ways of thinking was found to contribute to children's developing conceptions of geometrical concepts and their properties, including angle (as a turn or as shape) and reflective symmetry (Kaur, 2020; Ng & Sinclair, 2015).

3.2.4 Enhancing and assessing geometry learning

A significant body of literature focuses on approaches and didactic tools to support children's geometrical thinking and learning. A learning approach that has been studied by a few studies is play (e.g., Nakawa, 2020). Findings suggest that using children's play as starting points teaching mathematical content supports children's explorations of shapes (Bäckman, 2016), while guided play enhances children's shape knowledge (Fisher et al., 2013).

In addition, using picture book reading with (McGuire et al., 2021) or without the inclusion of additional mathematical activities (Van den Heuvel Panhuizen et al., 2015) has been found to be a promising avenue to contribute to the development of children's understanding of shapes and spatial relationships. However, in Nurnberger-Haag's (2017) study which examined the geometrical content of children's books, it was found that picture books often misteach shapes by using for example incorrect 2D names for 3D images and also by giving inaccurate properties or definitions.

Considering that spatial abilities are malleable (e.g., Bruce et al., 2017), a number of studies have focused on how spatial reasoning development in the early years can be supported. Various factors, including semiotic tools (Thom, 2018), classroom activities provided by the teacher (Bruce & Hawes, 2015) and contexts beyond school (Casey et al., 2014; Verdine et al., 2017) were found to play a significant role. Furthermore, spatial intervention studies within school demonstrate gains in children's spatial abilities, including spatial language, visual-spatial reasoning and 2D mental rotation (Bruce & Hawes, 2015; Cheng & Mix, 2014).

Research on the assessment of children's geometrical thinking and learning is rather scarce. The findings of Thom and McGarvey's (2015) study showed that drawing serves as a means to access, assess, and attend to children's understanding. Of interest is that children's spatial skills can be assessed very early. For example, a spatial skill test including spatial assembly, namely, Test of Spatial Assembly (TOSA) (Verdine et al., 2014) can be used already at the age of three, while a measure assessing mental rotation with tangible objects can identify developmental differences from the age of four up to the age of eight (Hawes et al., 2015).

3.2.5 Individual differences associated with learning in geometry

Research findings on the relationship between children's gender or socioeconomic status (SES) and geometrical learning are ambiguous. More specifically, Verdine et al. (2014) reveal that three-year-old children's performance in spatial assembly tasks did not differ by gender, while Jirout and Newcombe (2015) argue that boys have an advantage on the spatial skills compared to girls, because the boys spend more time with spatial play. The results of Milburna et al.'s (2019) study also showed no differences for geometrical abilities between males and females at the preschool age. Regarding socioeconomic background, lower SES children were found to lag behind higher SES children in spatial assembly skills (Verdine et al., 2014), while other studies (e.g., Jirout & Newcombe, 2015; Rittle-Johnson et al., 2017) reveal that children from low-income backgrounds follow a similar trajectory in geometry development as their peers from more advantaged backgrounds.

Another child-related factor that has been studied in relation to geometry learning is language abilities. Particularly, a number of studies suggest that first grade children's abilities in language comprehension and production as well as knowledge of geometrical words (e.g., of shapes) play a role in geometrical understanding (Mushin et al., 2013).

3.2.6 Concluding remarks

The growing body of research and the relevant evidence already provided for the importance of the embodied, spatial and dynamic aspects of early geometry learning indicate the need for generating and providing evidence for new frameworks in early geometry thinking, learning and development that would consider these characteristics and improve and deepen our knowledge about how children think, build and develop geometrical understandings and thus increase our awareness of the children's mathematical strengths and needs in this content area. For example, how spatial reasoning is associated with and could contribute to early geometry learning and to the development of geometrical thinking and understanding is a noteworthy issue (e.g., Dindyal, 2015) to be investigated systematically in future research.

The use of multimodal approaches (e.g., Calero et al., 2019) would play a crucial role in accessing and assessing children's understandings and thinking (e.g., drawings, oral speech, body movements, gestures, concrete objects) in the classroom (e.g., Thom, 2018). Besides, finding ways to assess effectively young children's geometrical understandings for learning is an important issue for further research, that was indicated by our findings.

Regarding shape knowledge and understandings, the findings of this survey show that more research needs to be undertaken for investigating children's competences and learning of 3D shapes. Furthermore, with respect to 2D shapes, children in recent studies still encounter challenges and difficulties (e.g., regular use of visual prototypes) (e.g., Kalénine et al., 2013) similar to those that have been revealed in earlier research (e.g., Levenson et al., 2011). The possible causes for this phenomenon need to be investigated more systematically. For example, it could be that there is a gap between research-based evidence/knowledge and educational policy, practice (e.g., teacher education and professional development, teaching) and curriculum development which needs to be addressed. Also, it is suggested that more research is needed on teaching strategies in geometry to support children move into more abstract ways of thinking (e.g., operational definitions of shapes, properties, shape relations). Particularly, based on our findings, the use of technologies, spatial programs, embodiment, play and picture books (e.g., Bäckman, 2016; Hallowell et al., 2015; Kaur, 2020; Van den Heuvel Panhuizen et al., 2015) offer a promising avenue towards this kind of learning.

Focusing on the individual differences associated with geometry learning, our findings revealed ambiguous results (e.g., Jirout & Newcombe, 2015; Verdine et al., 2014). The variability in the research conditions and procedure or educational system and culture from one study to another could provide an interpretation for this finding. A more systematic review or metanalysis on this specific issue and perhaps more comparative and cross-cultural studies for the domain of early geometry could give further insights into this thread.

3.3. Children's competencies in other content domains in early childhood

In this study, by other content domains in ECME we mean the mathematical content areas except for number sense and whole number development, and geometrical knowledge and skills, namely, patterns, measurement, spatial reasoning and thinking, reasoning and data modeling. A common focus of the literature reviewed across the different content domains is twofold: Firstly, providing insights into young children's competences and development and secondly, proposing interventions and investigating their effectiveness on children's learning.

3.3.1 Patterns

Patterns has been the focus of a large proportion of the reviewed literature on this theme (other content domains). Various studies are about pattern structure, visual patterns, repeating patterning competencies, etc. Seminal research on patterns in early childhood education was realized in the work of Mulligan and Mitchelmore (2013) who proposed Early Awareness of Mathematical Pattern and Structure (AMPS), and have shown that it generalizes across early mathematical concepts. This study provided a rationale for the construct of AMPS based on students' levels of structural development through five levels reliably categorized. A recent study explored how children can develop connected mathematical knowledge leading to generalization by developing patterns and structural relationships and showed the approach's effectiveness in modeling and expression, visualization and abstraction, and learning maintenance (Mulligan et al., 2020).

Furthermore, Björklund and Pramling (2014) clarified the importance of early childhood education activity in mathematics on the concept of 'pattern.' Also, Rittle-Johnson et al. (2015) illustrated the experiences preschool children receive with patterns and how their pattern knowledge changes over time. Regarding strategies for visual repeating patterns, Collins and Laski (2015) found that preschoolers completed a range of patterning tasks that varied in the extent to which they required mental representation and manipulation of the repeating unit. Preschoolers tended to use an appearance matching strategy on duplicate and extended tasks and a relational similarity strategy on transfer tasks. Tsamir et al. (2017) examined children's recognition of the unit of repeat and the structure of the repeating patterns and found that children can choose appropriate continuations which extend a repeating pattern beyond just one element.

A growing body of research has focused on children's recognition of the unit of repeating and the structure of the repeating patterns and the relations with other mathematical contents, such as numbers and arithmetic, algebra, calculation, or geometrical thinking (e.g., Hunter & Miller, 2022; Swoboda & Vighi, 2016; Venkat et al., 2018, etc.). It is shown that the patterning instruction was effective for recognizing symmetrical patterns, patterns with increasing numbers of elements, and patterns involving the rotation of an object (Kidd et al., 2014; Lüken & Kampmann, 2018, etc.). Furthermore, Bäckman (2016) proposed teaching and learning mathematical content through play to teach and learn shapes and patterns in four Swedish preschools.

Wijns et al. (2019) found that the ability to recognize the structure of patterns and understand mathematical language were strong predictors of their mathematics success, with the latter making a more significant contribution. Miller (2019) suggested that cross-curriculum opportunities in STEM education are the introduction of computer science as a fundamental skill/literacy for all students by using coding to identify mathematical structures and patterns.

3.3.2 Measurement

The following most significant number of publications was found on measurement. In measurement, length, area, mass, and time have received more attention, while only a few research papers are about bulk, volume, or weight. In a large part of these studies, the focus has been on length measurement. Kotsopoulos et al. (2013) showed the effects of different pedagogical approaches on kindergarten children's learning of length measurement. They evaluated and elaborated on the developmental progression or levels of thinking. Moreover, Szilágy et al. (2013) validated that the sequence of thinking levels in a hypothesized learning trajectory is consistent with observed behaviors of students from pre-kindergarten through second grade. Sarama et al. (2021) evaluated a part of their proposed learning trajectory, focusing on the instructional component, and found that instruction successfully promoted the children's progression. There are fewer pieces of literature about area measurement. Clements et al. (2018) verified the effects of instructional interventions through each level of a learning trajectory designed to support young children's understanding of area measurement as a structuring process.

3.3.3 Spatial reasoning

Research on spatial reasoning has rapidly increased in recent years, possibly because it has been elucidated that spatial reasoning is deeply related to later children's mathematical skills and may also lead to the development of proficiency in other subjects and fields besides mathematics. Reviewing spatial reasoning research brings together international literature across mathematics education, development, and cognition.

Extensive research has provided evidence for the positive relationship between spatial reasoning and mathematics competences and learning in the early years with emphasis on number-related abilities (e.g., Möhring et al., 2015; Van den Heuvel-Panhuizen et al., 2015). For example, Verdine et al. (2017) suggested that there are links between spatial reasoning and number-line estimation and counting in children already from age 3. These links between spatial and number-related skills could be explained by the mental number line and the linearity of mental representations of magnitude.

Kotsopoulos et al. (2021) investigated the overall verbal and nonverbal visual-spatial ability of 61 (34 boys) three- to five-year-olds and the following factors known to be related to visual-spatial ability: grade, sex, socio-economic status, math, and spatial activity engagement at home, parental mental rotation, quantitative reasoning, intelligence, and working memory. Results revealed that quantitative reasoning and general intelligence were significant predictors of overall and nonverbal visual-spatial ability. Mathematics activities in the home also predicted children's verbal visual-spatial ability.

The reverse relationship between spatial reasoning and mathematical abilities has been also studied in recent research primarily for the early primary school years and less for the preschool or kindergarten years. Evidence has been provided for the potential of early spatial reasoning programs in the classroom or in other contexts to significantly support mathematics learning (e.g., Cheng & Mix, 2014; Resnick, 2020; Woolcott et al., 2022; Zhang & Lin, 2017). In a synthesis of studies which investigated how spatial reasoning interventions contribute to mathematics learning in school, Woolcott et al. (2022) highlighted the importance of designing and evaluating spatial reasoning programs for primary school children to improve students' mathematics classroom learning, including evidence from standardized tests as they progress through the school system. Of interest is the focus of some studies on girls' (6–7-year-olds) early spatial and arithmetic skills. Particularly, Casey et al. (2014) and Laski et al. (2013) provided evidence for the importance of the development of early spatial skills on girls' effective mathematics learning.

Using another perspective, Gold et al. (2021) examined associations between engineering play with wooden unit blocks and mathematics and the spatial skills of children with and without disabilities. Findings provide initial evidence that engineering play is related to mathematics and spatial development and maybe a fundamental educational approach for supporting cognitive skills and school readiness in typically developing children and children with disabilities.

3.3.4 Thinking, reasoning and data modeling

Research in ECME has dealt with various kinds of thinking and reasoning. Functional thinking is one of the topics which several researchers investigated. Blanton et al. (2015) empirically developed a learning trajectory in first-grade children's (6-year-olds') thinking about the generalization of functional relationships by proposing the levels of sophistication in 6-year-olds' thinking about generalizing algebraic relationships in function data. Lenz (2022) explains that relational thinking and dealing with variables are two essential aspects of algebraic thinking. Examining the relational thinking of kindergarten and primary school children showed that specific conceptualizations of variables were related to children's ability to show relational thinking. In addition, Bakker et al. (2014) clarified children's informal knowledge of multiplicative reasoning. Even when assessed in a relatively formal setting, first-graders display a substantial knowledge of multiplicative reasoning before being taught.

English and Crevensten (2013) explored data modeling with a specific focus on structuring and representing data, including the use of conceptual and meta-representation competence, informal inference, and the role of context through the longitudinal study of data modeling in grades one to three. Supply et al. (2021) examined children's numerical abilities in the second grade of preschool and their probabilistic reasoning abilities one and two years later. They provided evidence for the predictive role of early numerical abilities on later probabilistic reasoning.

Kieran et al. (2016) investigated the nature of the research carried out in early algebra and how it has shaped the field's growth for the younger student aged from about six years to twelve years. This study found that mathematical relations, patterns, and arithmetical structures lie at the heart of early algebraic activity, with noticing, conjecturing, generalizing, representing, justifying, and communicating central to students' engagement. Obersteiner et al. (2015) investigated primary school children's use of strategies in contingency table problems and found that ignoring relevant information and referring to additive rather than multiplicative relationships between cell frequencies were among the children's primary strategies.

3.3.5 Concluding remarks

According to this review results, we showed the strengths in this current research body which focused on other content domains in mathematics in early childhood contexts. Children engaged in early childhood education should attain the essential knowledge and skills for learning and for the development of their understanding and skills in various mathematical content domains, such as patterns, measurement, algebraic thinking, spatial skills and data analysis. As children's early knowledge affects them for several years thereafter (Sarama & Clements, 2009), young children may need knowledge and skills drawn in various mathematical areas for their later learning of higher mathematical and other disciplinary contents, and basic knowledge and fundamental skills essential for human life in the future.

More extensive future research is worth pursuing in these content areas in ECME because the research questions addressed until now are quite limited. For example, systematically investigating the connections of early learning in these content domains with later mathematical performance at school is worth considering. Additionally, we recommend that further research could be carried out in the following topics: the nature of classroom culture and the role of the teacher, the forms of curricular activity, teaching strategies and new technologies on the learning of the above mathematical content areas, the links between children's competences in content areas beyond numeracy and their spatial and other cognitive skills and affective characteristics, as well as the contribution of spatial reasoning interventions on children's mathematical abilities in preschool.

3.4. Teaching and learning mathematics in early grades with technology-integrated activities

Although most of the research on the use of technology in mathematics education focused on older students, recent literature (2012-2022) also included reports on various uses of technology-integrated activities in early grades. These reports focused on different categories/threads, that we group and present in this paper (see Table 2) divided into the following sections.

3.4.1 Design features

A large body of literature focused on design features of different technological tools (e.g., spreadsheets, IWBs, dynamic geometry software, programmable toy robots, interactive applets for tablets, virtual manipulatives) used in early childhood mathematics education. Overall, such literature suggests that the use of modern technology supports students' learning, although the type and extent of such positive influence seem to depend on the age of the students (e.g., Moyer-Packenham et al., 2015). For example, an analysis of the movements and attitude of 3-4 year-old students during clinical interviews shed light onto differences in their relationships with touch devices and in the exploitation of such devices' affordances. Some studies showed that sometimes children can become even frustrated, in which case scaffolding could improve their learning (e.g., Bullock et al., 2017). However, most of the studies highlighted how students are attracted by the devices employed and this had positive repercussions on their attitude towards mathematics. Moreover, children seemed to be supported in incrementally refining their understanding and shaping their concept images of mathematical ideas (Watts et al., 2016).

Within the theme of design, two main lines of research can be outlined. The first one points out the need for designing new apps and virtual manipulatives for learning mathematics to create constructive opportunities to represent mathematical objects so that they can be manipulated (Moyer-Packenham, 2016). The second one highlights the importance of well-founded guidelines on the basis of this design, since there are many educational apps, and most are not designed following such guidelines. Moreover, researchers highlighted the need for more research on educational games and applications designed for preschoolers to establish whether, how, and for whom screen media can be educationally valuable. On the other hand, an under-informed approach not only disempowers teachers, but it also demotivates children. Guss et al. (2022) have taken a cross-sectional position advocating for a lens of equity, inclusion, and accessibility in the development of technology-based resources for early math using learning and teaching with learning trajectories.

Some studies focused on multi-touch devices, which seem to have the potential of supporting the development of number sense, particularly in low-performance students. Many of such studies were focused on the iPad app TouchCounts, which takes advantage of the multi-touch and gestures functionalities of the device and provides multimodal visual and auditory feedback for every touch or gesture. These kinds of simultaneous feedback can assist in the child's development of ordinality, cardinality, counting, adding and subtracting. Studies have focused on task design in TouchCounts, and the analyses of children's interactions have shown how collaborative practices can foster children's engagement and help develop their number sense (Baccaglini-Frank et al., 2020). In such a context children's shared gestures can become a material resource for making sense of numbers. There are examples of multiplayer learning environments involving embodied interactions, studied as new opportunities for training on the number line: improvements were found not only in accuracy on number line tasks but also in other numerical and arithmetic tasks (e.g., Baccaglini-Frank & Maracci, 2015; Sinclair, Chorney et al., 2016; Sedaghatjou, & Campbell, 2017).

A different but quite significant trend concerns the literature about robotics education; it includes studies that used robotics kits for young children or a single robot for a class of children in preschool or at the beginning of primary school (e.g., Bartolini Bussi & Baccaglini-Frank, 2015; Del Zozzo & Santi, 2023). Part of these studies focused on the impact of such activities with toy robots on visuospatial reasoning. For example, Di Lieto et al. (2017) provided support towards the hypothesis that educational robotics can progressively improve students' abilities in planning and controlling complex tasks even in early childhood. Francis et al. (2017) indicated the feasibility of designing robotic-enhanced activities for fostering STEM education in kindergarten. In particular, spatial reasoning seemed to play a vital role in choosing and being successful in STEM careers.

Table 2. Categories of the technology theme from reviewed papers and selected publications[a]

Category	Number of selected publications included in the paper	Sample of publications
Design features	13	Baccaglini-Frank & Maracci (2015); Baccaglini-Frank et al. (2020); Bartolini Bussi & Baccaglini-Frank (2015); Bullock et al. (2017); Del Zozzo & Santi (2023); Di Lieto et al. (2017); Francis et al. (2017); Guss et al. (2022); Moyer-Packenham et al. (2015); Moyer-Packenham (2016); Sedaghatjou & Campbell (2017); Sinclair, Chorney et al. (2016); Watts et al. (2016)
Technological tools to enhance and assess mathematics learning	4	Aunio & Mononen (2018); Axelsson & Andersson (2016); Baroody et al. (2014); Pazouki et al. (2018)
Pedagogical issues involved in teaching using technological tools	3	Bourbour & Masoumi (2017); Trgalová & Rousson (2017); Walshaw (2012)
Mathematics in focus: (mostly) number sense	10	Baccaglini-Frank & Maracci (2015); Baccaglini-Frank et al. (2020); Ferrara, & Savioli (2018); Holgersson et al. (2016); Rodney (2019); Rothschild & Williams (2015); Sedaghatjou & Campbell (2017); Sedaghatjou & Rodney (2018); Sinclair, Chorney et al. (2016); Wulandari et al. (2022)
Other issues	4	Bakos & Pimm (2020); Calvert et al. (2020); Hundeland et al. (2014); Sinclair & Heyd-Metzuyanim (2014)
n[b]	31	

[a] n=113 : Total number of papers analyzed

[b] Number of papers included in the paper (due to dual coding this number may be smaller than the sum of papers from all categories)

3.4.2 Technological tools to enhance and assess mathematics learning

Within this area of research, results are usually obtained by showing whether an experimental group of children improves its math skills significantly in comparison to a control group. A common research problem addressed by these studies is to understand the ways and means by which digital environments create a different and more effective (if so) learning experience of mathematical phenomena compared to environments characterized by more traditional media, such as paper and pencil.

A set of activities offered through educational software in the form of games designed to strengthen or remediate weaker abilities is frequently labelled "computer-assisted intervention". This kind of intervention has become increasingly common for young children, and the area of research is receiving growing interest and support (Axelsson & Andersson, 2016). Results suggested that game contexts act as motivators that can scaffold more mature cognitive capabilities in young children than they exhibit during a non-contextual standardized test (e.g., Aunio & Mononen, 2018).

In a study by Baroody et al. (2014), first graders participated in an experimental design using two computer-based programs developed to improve fluency in basic subtraction via guided learning strategies. The intervention succeeded in transferring skills and knowledge for subtraction.

Pazouki et al. (2018) discussed "MaGrid", a tablet-based application that provides a wide range of training tasks targeting fundamental mathematical concepts for the preschool level. The language-neutral property of MaGrid distinguishes it from other mathematical applications and it reduces the barrier of language for second language learners.

3.4.3 Pedagogical issues involved in teaching using technological tools

The primary goal in this strand of research seems to be understanding ways and means by which digital environments create a different learning experience of mathematical phenomena with respect to environments characterized by traditional media. Many of these studies explored the roles that preschool teachers give to technologies in mathematics education, and the ways in which they structure their activities when using technological artifacts (e.g., Walshaw, 2012).

The interactive white board (IWB) is relatively widely spread (e.g., Bourbour & Masoumi, 2017). In preschool it can be viewed as a multisensory resource to engage young children in reasoning and in problem-solving activities. As far as the teachers are concerned, having a negative attitude towards IWBs leads to a decrease in the likelihood of pedagogical change. Among the more positive aspects of the use of an IWB in first grade, studies listed: a dynamic display of the content, an increase in students' attention and motivation, having immediate feedback. Among the negative aspects they listed: technical difficulties, a frontal way of teaching, technical problems, a decrease in the teacher's control over students' work. It is noteworthy that studies investigating early childhood practicing or pre-service teachers employing tablets as part of their teaching practices in mathematics are scarce (e.g., Trgalová & Rousson, 2017).

3.4.4 Mathematics in focus: (mostly) number sense

Number sense has been the main mathematical focus within most of the technological environments studied in recent research. Also, set of studies on the use of robots focused on computing education. A minority of the studies reviewed focused on geometry, patterns or other mathematical content. Hence, we mainly describe the research on technology promoting number sense. Overall, the findings suggest that multi-touch technology has the potential to foster important aspects of children's development of number sense (e.g., Baccaglini-Frank et al., 2020; Ferrara, & Savioli, 2018; Rodney, 2019; Sedaghatjou & Rodney, 2018; Wulandari et al., 2022). In particular, there is evidence of learning while children are interacting with mathematics apps on touch-screen devices. Main findings also included the fact that different children attend to different affordances, suggesting that it is the key to offer appropriate scaffolding. As we discussed in the section 3.4.2, interacting within digital media contexts is quite complex for young children, so some studies have led to recommendations for interactive educational design as well as design trajectories to support the development of counting abilities and arithmetic competence in general (e.g., Rothschild & Williams, 2015).

An example of multi-touch app is Fingu, with the potential of enhancing children's speed and accuracy in subitizing and their ability to identify part-whole relationships (Baccaglini-Frank & Maracci, 2015; Holgersson et al., 2016).

More recent studies focused on preschoolers and primary school children engaged in learning numbers by interacting with TouchCounts (e.g., Sinclair, Chorney et al., 2016; Rodney, 2019). Some described the application, others investigated the mathematical, social and affective nature of children's engagement with the application, others focused on gestures both in their personal and social dimension, others yet analyzed the rhythm emerging from the interactions and some looked at students' learning of a specific mathematical issue related to numbers. For example, Baccaglini-Frank et al. (2020) and Sedaghatjou and Campbell (2017) explored how young students build an understanding of cardinality and ordinality principles through communicative, touchscreen-based activities involving talk, gesture and body engagement. These studies revealed that the implementation of open environments, like TouchCounts, supported child development of different strategies in response to tasks designed to address different number sense abilities. A substantial contribution is given by studies that deal with the nature and the role of gestures (e.g., Sinclair, Chorney et al., 2016). Finally, since gestures become multimodal resources to communicate temporal relationships about spatial transformations, a still emerging line of research concerns how to develop well-educated fingers in relation to engaging with mathematics.

3.4.5 Other issues

The literature review shows how technology-integrated activities in kindergarten and early grades may help improve not only competences such as mathematics or language but also transversal competences, such as communication, social interaction, equal education, co-responsibility with others, affectivity. Moreover, children make sense of the digital tools and are able to apply the tools purposefully as long as they also interact with an adult (e.g., Hundeland et al., 2014). For example, Sinclair and Heyd-Metzuyanim (2014) and Bakos and Pimm (2020) investigated the mathematical, social and affective nature of children's engagement with TouchCounts. When technology integration was accomplished successfully in early childhood education settings, children tended to interact more with one another, and exchange information related to computer tasks as well as to the overall classroom ongoing curriculum themes. In particular, Calvert et al. (2020) discussed that children's para-social relationships and para-social interactions with intelligent characters may provide new frontiers for 21^{st}-century learning.

3.4.6 Concluding remarks

From the literature review, we can conclude that well-planned integration of technology in the classroom, with clear learning objectives and appropriate feedback, can motivate children, enhance concentration, and support independent learning and communication. Mathematics apps could be fruitful tools also outside school. Indeed, one of the problems that is usually pointed out is that students rarely practice math outside of school requirements - called "math-practice gap" (Stacy et al., 2017) – and it may contribute to students' struggle with mathematics. Mathematics apps offer a viable solution to this problem, providing access to many problems, tied to immediate feedback, and delivered in an engaging way.

A common goal of the studies discussed was to explore effective use of apps in mathematics teaching and how such use enables the learning process, making learning meaningful and student-centered. By exploring the connections between mobile devices, media literacy and visual literacy, some studies emphasized the collaborative affordances of many apps and the importance of multimodal forms of representation fostered through gesture, voice, text, video and audio.

A still open problem is that teachers are frequently not confident about their ability to teach mathematics using technology (Dong, 2018). There is a difference in the use of various technological artifacts which can be attributed to their being either fixed or mobile. For example, the IWB does not seem to pose pedagogical challenges to teachers, as its stable location offers the opportunity of using it in traditional teaching ways. On the other hand, tablets seem to be a challenge for some teachers because of the need they pose to reconfigure the organization and the roles of teachers and students (e.g., Brown & Englehardt, 2017; Fekonja-Peklaj & Marjanovič-Umek, 2015). Further research is necessary to address these issues and explore prompting actions to support teachers in realizing the potential of technology and, at the same time, to design appropriate training and guidelines in ECME. Some university departments for early childhood preservice teachers include courses in effective ways of using ICT and integrating it into daily education, but these actions are not yet sufficient.

3.5. Early childhood teachers' knowledge, education and affective issues in mathematics

Teachers' knowledge and beliefs for teaching mathematics are complex and multifaceted. This is true also for early childhood teachers. This section begins by describing some theoretical conceptualizations of early childhood educators' knowledge and beliefs and then review studies which examined specific components of knowledge and beliefs. Finally, professional development (PD) studies are reviewed. The categories from the reviewed studies on this theme and the relevant publications are presented in Table 3. The terms preschool and kindergarten are used in accordance with the study being reviewed.

3.5.1 Theoretical models and frameworks: knowledge and competence

Nearly all frameworks and models of preschool teachers' knowledge relate to what is termed subject-matter knowledge (SMK) and pedagogical-content knowledge (PCK). For preschool teachers, mathematical content refers to the concepts and skills mentioned by curricula, such as numbers and operation, measurement, geometry, data representation, and patterns (Ren & Smith, 2018). Gasteiger and Benz (2018) added that preschool teachers should also know the structure of mathematical concepts so that early mathematics education may be implemented coherently. The Cognitive Affective Mathematics Teacher Education (CAMTE) framework (e.g., Tsamir et al., 2014) differentiated between two components of SMK – being able to solve mathematical problems and being able to evaluate solutions of mathematical tasks. PCK may also be broken down into sub-components, such as knowledge of young children's mathematical conceptions, and knowledge of appropriate mathematical tasks (e.g., Tsamir et al., 2014).

Combining SMK and PCK into one model, Gasteiger and Benz (2018) used the term explicit knowledge to include knowledge of mathematical concepts and knowledge of developmental processes. They also added situational observing, pedagogical actions, and evaluation. They theorized that knowledge of development processes would inform teachers' ability to assess the learning level of students, which in turn would affect teachers' actions.

Table 3. Categories of the teachers' theme from reviewed papers and selected publications[a]

Category	Number of selected publications included in the paper	Sample of publications
Theoretical models and frameworks: knowledge and competence	3	Gasteiger & Benz (2018); Ren & Smith (2018); Tsamir et al. (2014)
Teachers' knowledge for teaching preschool mathematics	19	Benz (2016); Björklund (2012); Canturk-Gunhan & Cetingoz (2013); Dunekacke et al. (2016); Gasteiger & Benz (2018); Lee (2017); Lembrér et al. (2018); McGarvey (2012); Moss et al. (2015); Opperman et al. (2016); Paolucci & Wessels (2017); Paparistodemou et al. (2014); Schack et al. (2013); Tanase & Wang (2013); Tirosh et al. (2019); Torbeyns et al. (2020); Tsamir et al. (2014); Tsamir et al. (2015); Ulusoy (2021)
Affective issues related to teaching mathematics during the early years	17	Benz (2012); Can & Durmaz (2023); Cross Francis (2015); Dunekacke et al. (2016); Franzén (2014); Gasteiger & Benz (2018); Jenßen et al. (2022); Li et al. (2019);Opperman et al. (2016); Polly et al. (2018); Ren & Smith (2018); Russo et al. (2020); Sancar-Tokmak (2015); Sumpter (2020); Theil & Jenssen (2018); Tsamir et al. (2013); Tsamir et al. (2014)
Professional development	10	Cross Francis (2015); Ertle et al. (2016); Gasteiger & Benz (2018); Heng & Sudarshan (2013); Moss et al. (2015); Olfos et al. (2022); Polly et al. (2018); Tsamir et al. (2014); Wilson et al. (2013); Wullschleger et al. (2023)
n[b]	40	

[a] n=54 : Total number of papers analyzed

[b] Number of papers included in the paper (due to dual coding this number may be smaller than the sum of papers from all categories)

3.5.2 Teachers' knowledge for teaching preschool mathematics

While numerous studies have explored the interrelationships between various dimensions of knowledge, as well as their connections to pedagogical competencies and beliefs (e.g., Opperman et al., 2016), others have directed their attention towards specific facets of content knowledge. Within the domain of patterns, McGarvey (2012) examined how teachers classify images as patterns or not patterns. A frequent criterion for an image to be considered a pattern was if an element of repetition could be recognized, and if the person could tell what would go next. In the domain of geometry, Tsamir et al. (2015) reported that teachers could identify examples and non-examples of triangles and define triangles; identify examples and nonexamples of circles but had difficulty defining circles; and had some difficulties in both identifying examples and non-examples of cylinders and defining cylinders. Similar results were found among prospective preschool teachers in Turkey (Canturk-Gunhan & Cetingoz, 2013). Ulusoy's (2021) research suggested that prospective teachers were not aware of the potential influence of using triangle-looking non-examples when teaching about triangles. Moss et al.'s (2015) research suggested a need to promote teachers' knowledge of dynamic aspects of geometry. It should be noted that studies investigating teachers' content knowledge of measurement, data and combinatorics were scarce.

Studies that concerned teachers' PCK were more frequent than studies of teachers' content knowledge. The methodology of these studies varied and included quantitative methods, such as handing out questionnaires (e.g., Dunekacke et al., 2016), qualitative observations of teachers' lessons (Björklund, 2012), and qualitative analysis of teachers' noticing and attending to written scenarios (Oppermann, et al., 2016), videos (Schack et al., 2013), and pictures (e.g., Lembrér et al., 2018). Of note is the "Preschool Mathematics PCK interview", a tool developed by McCray and Chen (2012), which comes in the form of a vignette describing a free play situation serving as the basis for an interview, guided by questions which ask teachers to identify the mathematics embedded in the situation and to propose ways to support children's awareness of the particular mathematical topic and experimentation with it in their play.

Several studies focused on teachers' abilities to recognize mathematical situations that occur during children's natural play (Benz, 2016; Björklund, 2012) or during play-based scenarios (Lee, 2017; Oppermann et al., 2016; Torbeyns et al., 2020). Noticing children's early numeracy was also investigated among prospective elementary school teachers (Schack et al., 2013). In that study, three noticing skills were analyzed: attending, interpreting, and deciding. Gasteiger & Benz (2018) pointed out that a teacher who purposely observes situations and specifically looks for mathematical relevance is more likely to make appropriate pedagogical decisions than a teacher who does not notice or look for the mathematics involved in situations.

Related to PCK is knowing what types of examples and manipulatives to employ in mathematical activities. Within a geometrical context, Paparistodemou et al. (2014) found that most prospective teachers considered only prototypical shapes in their tasks. Within the domain of patterns, Tirosh et al. (2019) investigated preschool teachers' examples of repeating patterns. Paolucci and Wessels (2017) investigated prospective teachers' capacity to create mathematical modeling problems for grades one to three.

Finally, when planning and implementing mathematical activities, it is important to consider students' conceptions, strategies, developmental trajectories, skills, and common errors (Tanase & Wang, 2013). Although the mathematical content of early childhood includes several domains, this line of study was pursued mostly regarding knowledge of students' number conceptions and strategies. For example, preschool teachers were found to overestimate children's ability to verbally count to 30 and underestimate children's ability to say what number comes right after six (Tsamir et al., 2014).

3.5.3 Affective issues related to teaching mathematics during the early years

Teachers' beliefs and attitudes towards teaching mathematics in early childhood were also investigated. Regarding the relevance of mathematics to preschool, studies found that prospective and practicing preschool teachers may hold negative attitudes, and believe that engaging in mathematics is less important than engaging in language and arts. However, with further education, these beliefs become more positive (Sumpter, 2020). On the other hand, other studies found that practicing kindergarten teachers do recognize the importance of learning mathematics in kindergarten (Benz, 2012) and engage children with mathematics during everyday situations. Thus, we see a connection between beliefs and practice. Regarding appropriate mathematical content, kindergarten teachers mostly consider counting as very relevant (Cross Francis, 2015), and more important than geometry, measurement (Benz, 2012). Additional studies investigated teachers' beliefs related to the nature of mathematics (Dunekacke et al., 2016; Jenßen et al., 2022) finding that such beliefs may be divided into three sub-constructs: a static orientation of mathematics in which mathematics is seen as a clear system of rules, a processes orientation, and an application orientation. The last two orientations view mathematics as dynamic in nature.

Concerning pedagogical mathematical beliefs, teachers stress children's need to use their bodies as tools for learning mathematics, for example by climbing up and down to feel differences in height (Franzén, 2014). Teachers also believe that it is essential for children to be active learners (e.g., Benz, 2012) and that the teacher's roles are to ask questions (Cross Francis, 2015), instill curiosity, and encourage children to think by themselves (Li et al., 2019). Preschool teachers, more than lower elementary school teachers, believe that one way to encourage thinking and increase interest in mathematics learning is by integrating children's literature into mathematics education (Can & Durmaz, 2023). Lower elementary school teachers believe that students need to learn perseverance in the face of problems they cannot immediately solve (Russo et al., 2020). As with knowledge, PD can affect teachers' beliefs (Polly et al., 2018).

Self-related beliefs, such as confidence, motivation, anxiety, self-concept, and self-efficacy were also investigated. Ren and Smith (2018) found links between teachers' contextual factors, such as professional backgrounds, and teachers' confidence in learning mathematics, mathematics anxiety, and motivation to learn mathematics. For example, higher mathematics knowledge for teaching was associated with lower mathematics anxiety. Mathematics anxiety may also cause teachers to avoid mathematics-related courses, even when those courses are geared for early childhood teachers (Theil & Jenssen, 2018).

Theil and Jenssen (2018) and Opperman et al. (2016) investigated preschool teachers' mathematical self-efficacy, that is, their confidence to solve mathematical problems. In general, teachers' mathematical self-efficacy was above the theoretical mean, while the self-concept was below the theoretical mean. Investigating prospective preschool teachers' mathematics teaching self-efficacy (i.e., their belief in their ability to teach mathematics), Sancar-Tokmak (2015) found a significant increase in participants' mathematics teaching efficacy at the end of a PD program. Several studies investigated preschool teachers' self-efficacy related to specific mathematical content, such as geometry or patterning, as well as to specific pedagogical tasks (e.g., Tsamir et al., 2013).

Many educators agree that preschool teachers' knowledge and beliefs are related. Specifically, Gasteiger and Benz (2018) pointed out that how teachers use their knowledge and skills can be influenced by their beliefs, attitudes, and motivation. Mathematical PCK and the application–related orientation of mathematics (i.e., mathematics is applied in real life situations) can predict noticing skills (Dunekacke et al., 2016). In another study, it was found that higher knowledge was associated with more positive attitudes and higher levels of student-centered beliefs (Ren & Smith, 2018). It could also be that mathematical beliefs impact on teachers' acquisition of mathematical knowledge for teaching. Finally, a longitudinal study covering the transition period between training and becoming a teacher, revealed that while knowledge increases over time, most beliefs do not change (Jenßen et al., 2022).

The relation between self-beliefs and knowledge was also theorized and investigated. Opperman et al. (2016) hypothesized that preschool teachers' mathematical content knowledge and their mathematical self-efficacy and self-concept are interrelated and that together they would affect preschool teachers' sensitivity to mathematics in play-based situations. Their study confirmed that mathematical content knowledge predicted sensitivity to mathematics in children's play. However, mathematical ability beliefs were found to serve as a filter. Other studies investigated if preschool teachers' self-efficacy beliefs were in line with their actual knowledge (Tsamir et al., 2014).

3.5.4 Professional development

When surveying studies related to the mathematics education of early childhood teachers, few studies related specifically to the preparation of prospective teachers to teach mathematics in preschool. One exception is Olfos et al. (2022) who pondered the difficulty of bridging theory with practice for future preschool teachers. They proposed a teacher training device called the Scaffolding System, which included two components: conceptual content for teaching mathematics to young children (e.g., counting, number composition and decomposition), and teacher training strategies such as video analysis and Lesson Study. Their study of 170 third-year early childhood education students in Chile found that the connection between theory and practice was felt most in the process of lesson preparation.

Most studies investigated PD for practicing teachers, which at times included in the same program, or in a parallel program, prospective teachers (e.g., Wilson et al., 2013). In general, papers surveyed for this study focused on promoting teachers' knowledge of their young students' mathematical thinking. Some programs centered on specific mathematical content, such as promoting teachers' knowledge of students' number conceptions and counting skills (Polly et al., 2018; Tsamir et al., 2014) or increasing teachers' appreciation for children's capacities to learn geometry (Moss et al., 2015). In general, however, the main message of these papers was the importance of increasing teachers' awareness of students' thinking, along with helping teachers develop and use tools to better understand children's engagement with mathematics.

Of course, the aim of understanding children's thinking is to offer children appropriate opportunities for engaging with mathematics. Towards this aim, a recent study (Wullschleger et al., 2023) compared the effectiveness of two programs for preschool teachers, one focused on teacher-child interactions (micro-adaptive learning) and the other focused on planning, preparation, and reflection (macro-adaptive learning). Both programs had a small positive effect on the issues it was designed to improve. The authors concluded that the two types of adaptive support are different and thus require different professional development courses for each.

Different programs used different methods to increase teachers' knowledge of their students and how to interact with young children and mathematics, but nearly all had some practical element. Many programs encouraged teachers to engage children in their preschool classes with activities, usually, but not always, developed during the program, video these encounters, and then reflect on them (Gasteiger & Benz, 2018; Tsamir, et al., 2014; Wilson et al., 2013). Other studies employed video examples (Cross Francis, 2015; Ertle et al., 2016), role play (Wullschleger et al., 2023), or made use of learning trajectories to support teachers' making sense of students' thinking (Wilson et al., 2013). Several researchers suggested employing the clinical interview method to foster a deeper understanding of children's thinking, to carry out formative assessment, and to plan further instruction (Heng & Sudarshan, 2013; Ertle et al., 2016; Polly et al., 2018).

3.5.5 Concluding remarks

A common thread running throughout this part of the survey is the emphasis on children. More studies of teachers' knowledge focused on teachers' PCK than on their SMK, and within PCK, research focused on coming to know young students as mathematics learners. Studies of teachers' beliefs investigated beliefs regarding the relevance of mathematics for young children, and what is appropriate mathematics for children. Finally, many PD programs focused on enhancing teachers' knowledge of children's mathematical abilities and reasoning.

This review also highlighted some gaps. Few studies (e.g., McGarvey, 2012; Moss et al. 2015; Tsamir et al., 2015) focused on preschool teachers' SMK. Those that did, showed teachers to be more knowledgeable in some areas than in others. Further research could help teacher educators plan for professional development in additional content domains, such as data representation and measurement. One challenge of professional development is scaling up. Another area in need of research is teachers' knowledge and beliefs related to the use of technology in preschool mathematics. Finally, few studies specifically investigated interventions for prospective preschool teachers (Olfos et al., 2022). This is an important area of research that could help us improve mathematics education for prospective preschool teachers, increasing their knowledge, along with their self-efficacy and motivation, to engage their young students with mathematics.

4. Conclusion

In the past few years, we gained considerable knowledge about the learning and teaching of mathematics in early childhood across the five themes that we identified in our survey.

Regarding the mathematics observed in children, based on the content-related themes of the review, our findings demonstrated that the strongest emphasis on research is in number sense and development. Papers examined children's competences in various aspects of number development, including counting, approximate number system, SFON, and proposed and evaluated strategies to enhance children's numerical knowledge and skills. A smaller amount of research focused on geometry education in the early years. Within this domain, much research attention is given to the embodied, spatial and multimodal approaches in early geometry, which were found to play a crucial role in children's geometrical learning and thinking.

Papers on other content domains investigated competences in relation to patterns, measurement, spatial thinking, reasoning and data modeling. A greater amount of this research focused on spatial thinking and patterns which provided evidence for the links between children's competences in each of these domains with their abilities in other mathematical contents.

The reviewed literature of the technology theme indicated that technology in early mathematics takes different forms and has various design features, which can create different learning experiences of mathematics for young children compared to the use of more traditional methods, and can support children's learning and enhance their dispositions towards mathematics. Of note is that the main mathematical focus of the studies on technological environments that were analyzed has been number sense.

A considerable number of publications in ECME focused on issues related to EC teachers. Greater emphasis was given on their PCK and specifically with respect to children's mathematical thinking and learning rather than their SMK. Many studies focused on teachers' beliefs regarding the relevance of mathematics for children, and on teaching approaches of mathematics in the early years. Professional development programs in various studies mainly aimed to increase teachers' knowledge of children's mathematical thinking. A number of papers pointed to the links between teachers' knowledge, beliefs and professional development.

Overall, we know much more than a decade ago about what mathematics children know and (can) learn before or at the beginning of formal education, how beginning learners of mathematics learn mathematics and develop their mathematics skills, how to improve and enhance early mathematics learning and stimulate the development of children's mathematical competences and also on teachers' knowledge, acts and beliefs related to early years mathematics. However, there is much more to learn.

We expect greater research attention on assessing and developing children's competences in content domains beyond early numeracy, including geometry, spatial reasoning, pattern and structure, measurement, statistical reasoning, functional thinking. For example, the contribution of cognitive skills to children's learning in these mathematical domains is an underinvestigated issue which needs further consideration by ECME research. Substantial growth of research on the aforementioned mathematical content domains would generate new knowledge and improve existing knowledge which in turn would contribute to the promotion of these mathematical content strands in the curricula and teachers' education. Furthermore, young children have an informal knowledge of mathematics that is broad, rich and complex, as they engage in mathematical thinking and reasoning in various contexts, in which they explore patterns, spatial relations, compare magnitudes and so on (Seo & Ginsburg, 2004). Thus, from a practical point of view, considering that the possible knowledge gaps in children are to a large extent due to the absence of links between informal knowledge and school mathematics (Sarama & Clements, 2009), it is likely that ECME which promotes concepts and skills from various mathematical domains would address these gaps.

Our work indicates that, with respect to children's age, the focus of ECME research has been on kindergarten, prekindergarten and early primary school years. Considering that children begin to possess and develop mathematical abilities from their infancy, it is likely that various mathematical competences emerge and embed earlier than has been suggested by existing research. Thus, we hope to see further and deeper investigations of a broader range of mathematical skills (beyond numeracy) and development of children under four years of age, using appropriate research designs and procedures. More research is also needed on the learning opportunities in mathematics offered to toddlers and infants and on developing early childhood educators' knowledge about this age group of children as mathematics learners.

Furthermore, we expect to see continued growth in research on digital tools and particularly on how technology can be designed and used in early childhood settings to enhance or support children's learning in different content domains and on how to empower teachers in effectively using technology in early childhood mathematics.

Given the importance of embodied cognition in mathematics and based on the findings of our review on embodied learning in certain themes (e.g., geometry education, technology), it is pertinent to see more research on this topic focusing on how to offer children more opportunities to engage in embodied ways of mathematical thinking and learning (with or without the use of technology) in the classroom and the role of the teachers and teaching practices in this in different mathematical content domains. Play (either adult-initiated or child-initiated) has been the focus of papers across all the themes of this review. Many of these studies provided evidence for the benefits of play in children's mathematical development. This finding as well as the imperative role to play in children's lives highlights the need to be studied more systematically and in more depth in future early childhood mathematics research. What matters in supporting children's learning in mathematics seems to be the quality rather than the quantity of play. Future research is expected to focus on the mathematics that play stimulates in children, the qualities of play that foster children's mathematics learning, how these qualities vary in different contexts and the teachers' role in using play to support children's mathematical learning and development.

Finally, because of the broad scope of ECME research, the present survey, despite its comprehensive character, could not address all the aspects of this rich and dynamic field of study. Drawing on our research efforts, several key themes have emerged that warrant more systematic exploration in future research reviews within the field of early childhood mathematics education (ECME). These include the impact of contexts beyond the school environment, such as the home setting; the role of young children's emotional characteristics, like self-efficacy beliefs and spatial anxiety; and the integration of mathematics with other subjects like music and art, influencing children's mathematical learning and developmental pathways.

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