Some modelling issues about dependence coming from finance.

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and joint work with Martino Grasselli (Università di Padova and De Vinci Finance Lab, Paris la Defense) & Christopher Van Weverberg (ULB)

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18-19 Septembre 2014
Part 1: Modelling correlated processes in a multivariate Lévy framework

1. Introduction Part 1
2. Multivariate Lévy model
3. Quanto Futures
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Part 2: Modelling correlated processes by Wishart processes

1. Introduction Wishart processes
2. Moment Explosions
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Part1 : Modelling correlated processes in a multivariate Lévy framework

Joint work with Laura Ballotta (Cass Business School, City University London) and Grégory Rayée (ULB)
Introduction : Modelling correlated processes in a multivariate Lévy framework

- Examples of multivariate correlated processes in Insurance and Finance:
  - Dynamic dependence between interest rate processes, market prices (stock, real estate), longevity and inflation
  - Sub-portfolios versus a global insurance portfolio: the influence of the dependence upon the solvency capital
  - Counterparty credit risk in a multivariate structural model with jumps
  - Multi-dimensional derivatives like exchange options and quanto options

- Goal: Modelling of correlated processes in a multivariate Lévy setting with no mean-reverting properties

- In this talk we will introduce exchange options, but mainly concentrate upon quanto options.

- We will discuss historical correlation versus implied correlation.
Pricing Options on a single underlying asset $(S)$:

- **Example**: European Call with strike $K$ and maturity $T$
  \[ C(K, T) = e^{-rT} E^Q[(S(T) - K)^+]. \]

- Univariate Lévy model:
  \[ S(t) = S(0)e^{X(t)} \]

where $X = \{X(t), t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lévy process, namely,

- $X$ starts at 0: $X(0) = 0$ a.s.,
- $X$ has independent increments:
- $X$ has stationary increments:
  \[ \Rightarrow X(t) \text{ is infinitely divisible}. \]
It can be deduced from the above observation that any Lévy process has the property that for all $t \geq 0$, the characteristic function $\phi_{X_t}(u)$

$$\phi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] = e^{\varphi(u)t}$$

where $\varphi(u)$ is the characteristic exponent of $X_1$, which has an infinitely divisible distribution.

**The Lévy–Khintchine formula**: $X$ has an infinitely divisible distribution if and only if there exists a triple $(a, \sigma, \nu)$, where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and $\nu$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|^2)\nu(dx) < \infty$ such that,

$$\varphi(u) = iau - \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}_0} (1 - e^{iux} + iux\mathbb{I}_{|x|<1})\nu(dx)$$
The Black Scholes model (BS):

- $X^{BS}(t) = (\mu - \frac{\sigma^2}{2})t + \sigma B(t)$,
- $\{B_t, t \geq 0\}$ is a standard Brownian motion,
- $\sigma > 0$
  \[ \Rightarrow X^{BS}(t) - X^{BS}(0) \sim \mathcal{N}((\mu - \frac{\sigma^2}{2})t, \sigma^2 t) \]
- The characteristic function:

\[
\phi_{BS}(u; \mu, \sigma) = \mathbb{E}[e^{iuX^{BS}(t)}] = e^{(iu(\mu - \frac{\sigma^2}{2})t) - \frac{(\sigma u)^2}{2} t}.
\]
The Variance Gamma model (VG):


\[ X^{VG}(t) = \theta G(t) + \sigma B_G(t), \]

- \( \{B_t, t \geq 0\} \) is a standard Brownian motion,
- \( \sigma > 0 \) and \( \theta \in \mathbb{R} \),
- \( \{G_t, t \geq 0\} \) is a Gamma process

\[ f_{G_t}(x; \frac{t}{k}, k) = x^{t-1} \frac{e^{-\frac{x}{k}}}{\Gamma(\frac{t}{k})k^\frac{t}{k}}, \quad k > 0, \]

The characteristic function and the characteristic exponent:

\[ \phi_{VG}(u; \sigma, k, \theta) = \mathbb{E}[e^{iuX^{VG}(t)}] = \left(1 - iu k \theta + \frac{1}{2} ku^2 \sigma^2\right)^{-\frac{t}{k}}. \]

\[ \varphi_{VG}(u; \sigma, k, \theta) = -\frac{1}{k} \ln \left(1 - iu k \theta + \frac{1}{2} ku^2 \sigma^2\right) \]
Introduction Part 1

Univariate Lévy Model

Standard Brownian Motion

VG Process (C=20; G=40; M=50)
Introduction

Pricing Options written on more than one underlying asset:

- Example: **Exchange Option**
  
  \[ E(S_1, S_2, T) = e^{-rT} E^Q[(S_2(T) - S_1(T))^+] \].

- Exponential multi-Lévy

  \[
  S_1(t) = S_1(0)e^{Y_1(t)} \\
  S_2(t) = S_2(0)e^{Y_2(t)}
  \]

  where \( Y_1(t), Y_2(t) \) are dependent Lévy processes (BM, VG, NIG,...).

- Restrictions on the range of possible dependencies and the set of attainable values for the correlation coefficient.

Multivariate Lévy processes via Linear Transformation:

\[ S_1(t) = S_1(0)e^{Y_1(t)+a_1Z(t)}, \]
\[ S_2(t) = S_2(0)e^{Y_2(t)+a_2Z(t)}. \]

where \( Y_1(t), Y_2(t) \) and \( Z(t) \) are independent Lévy processes.

- The common process \( Z(t) \) can be considered as the systematic part of the risk
- The processes \( Y_1(t) \) and \( Y_2(t) \) can be seen as capturing the idiosyncratic shock


Calibration Method of Ballotta & Bonfiglioli (2014)


- Let assume that $X_1(t) = Y_1(t) + a_1 Z(t)$ and $X_2(t) = Y_2(t) + a_2 Z(t)$.

$$S_1(t) = S_1(0) e^{X_1(t)},$$

$$S_2(t) = S_2(0) e^{X_2(t)}.$$  

where $Y_1(t)$, $Y_2(t)$ and $Z(t)$ are independent Lévy processes.

- Specify univariate Lévy processes for $X_1(t)$ and $X_2(t)$.
- Calibrate $X_1(t)$ and $X_2(t)$ with respect to the Vanilla market.

$$\min_{\chi_1} \sum_{i,j} \left( C_{\text{mod}}(S_1, K^i, T^j) - C_{\text{mkt}}(S_1, K^i, T^j) \right)^2$$

$$\min_{\chi_2} \sum_{i,j} \left( C_{\text{mod}}(S_2, K^i, T^j) - C_{\text{mkt}}(S_2, K^i, T^j) \right)^2$$
Calibration Method of Ballotta & Bonfiglioli (2014)

Then choose $Y_1(t)$, $Y_2(t)$ and $Z(t)$ from the same family of processes as $X_1(t)$ and $X_2(t)$.

To determine $Y_1(t)$, $Y_2(t)$ and $Z(t)$ parameters, they impose the following conditions:

1. Convolution condition, namely, the linear combination $Y_1(t) + a_1Z(t)$ has the same given distribution of $X_1(t)$ and $Y_2(t) + a_2Z(t)$ has the same given distribution of $X_2(t)$.

   \[
   \phi_{X_1}(u) = \phi_{Y_1}(u)\phi_Z(a_1 u) \\
   \phi_{X_2}(u) = \phi_{Y_2}(u)\phi_Z(a_2 u) 
   \]

2. Correlation condition

   \[
   Corr(X_1(t), X_2(t)) = \frac{a_1a_2 Var(Z(1))}{\sqrt{Var(Y_1(1)) + a_1^2 Var(Z(1))}\sqrt{Var(Y_2(1)) + a_2^2 Var(Z(1))}}
   \]
Drawback of this Calibration Method:

- The convolution condition is quite restrictive. For example, a linear combination of two VGs is not necessarily VG.
- When the number of underlying assets is high, it becomes difficult to fit the correlation Matrix and the convolution conditions.
Multivariate Lévy model

- No convolution condition: \((X_1(t) \text{ and } X_2(t))\)

\[
S_1(t) = S_1(0)e^{(r-\bar{\theta}_1+\omega_1)t}e^{Y_1(t)+a_1Z(t)}
\]

\[
S_2(t) = S_2(0)e^{(r-\bar{\theta}_2+\omega_2)t}e^{Y_2(t)+a_2Z(t)}
\]

where \(Y_1(t), Y_2(t)\) and \(Z(t)\) are independent Lévy processes and \(\bar{\theta}\) a dividend rate.

- \(Y_1(t), Y_2(t)\) and \(Z(t)\) can belong to any family of Lévy processes. Even a mix is possible, we can choose \(Y_1(t), Y_2(t)\) and \(Z(t)\) from different family of processes.

- The Mean-Correcting Martingale Measure \(Q\) with \(E^Q[S(t)]\) also denoted by \(E[S(t)]\)

\[
E[S(0)e^{((r-\bar{\theta})t+Y(t)+aZ(t))}] = S(0)e^{(r-\bar{\theta})t}
\]

- The risk neutral adjustment factor \(\omega\) is given by

\[
\omega = -\varphi_Y(-i) - \varphi_Z(-ai)
\]

In case of VG:

\[
\omega = \frac{1}{k_Y} \ln \left(1 - k_Y \theta_Y - \frac{1}{2} k_Y \sigma^2\right) + \frac{1}{k_Z} \ln \left(1 - ak_Z \theta_Z - \frac{1}{2} k_Z a^2 \sigma^2\right).
\]

- In the following we will assume a zero dividend rate \(\bar{\theta} = 0\).
Option pricing

**FFT Method:**

- The Carr-Madan type formula:

  \[ C(S, K, T) = e^{-rT}E[(S(T) - K)^+] \]

  \[ = e^{-\alpha \ln(K)} \int_0^{+\infty} e^{-i v \ln(K)} \psi(v) dv \]

  where

  \[ \psi(v) = \frac{e^{-rT}E[e^{i(v-(\alpha+1)i) \ln(S(T))}]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} = \frac{e^{-rT} \phi_{ln S}(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \]

  \[ \phi_{ln S}(u) = E[e^{iu \ln(S(T))}] \]

  \[ = E[e^{iu(ln(S(0)+(r-\bar{q}+\omega)T)+Y(T)+aZ(T))}] \]

  \[ = e^{iu(ln(S(0)+(r-\bar{q}+\omega)T)E[e^{iuY(T)}]E[e^{iuaZ(T)}]} \]
Pricing Exchange Option:

- The arbitrage free price of an Exchange Option $E(S_1, S_2, T)$ is given by

$$E(S_1, S_2, T) = e^{-rT}E^Q[(S_2(T) - S_1(T))^+]$$

$$= e^{-rT}E^Q[S_1(T)(\frac{S_2(T)}{S_1(T)} - 1)^+]$$

$$= S_1(0)E^{P^*}[(A(T) - 1)^+]$$

where $h = 1$.

- The pricing of the exchange option reduces to the pricing of a European Call Option on $A(T)$ under the measure $P^*$ with strike $K = 1$ and a maturity $T$.

- This is in fact the density process of an Esscher change of measure with $h = 1$. See e.g. Gerber and Shiu (1994), Hubalek and Sgarra (2006) and Eberlein, Papapantoleon and Shiryaev (2009).
The pricing of the exchange option reduces to the pricing of a European Call Option on \( A(T) \) under the measure \( \mathbb{P}^* \) with strike \( K = 1 \) and a maturity \( T \).

\[
A(T) = \frac{S_2(T)}{S_1(T)} = \frac{S_2(0) e^{(r-q_2+\omega_2)T+Y_2(T)+a_2 Z(T)}}{S_1(0) e^{(r-q_1+\omega_1)T+Y_1(T)+a_1 Z(T)}}
= A(0) e^{(\omega_2-\omega_1)T+(q_1-q_2)T+(Y_2(T)-Y_1(T))+(a_2-a_1) Z(T)}
\]

\( \textbf{FFT Method} : \) The Carr-Madan formula :

\[
E(A, K, T) = S_1(0) E^{\mathbb{P}^*}[(A(T) - K)^+] = \frac{e^{-\alpha \ln(K)}}{\pi} \int_0^{+\infty} e^{-i \ln(K) \rho(v)} dv
\]

where

\[
\rho(v) = \frac{S_1(0) E^{\mathbb{P}^*}[e^{i(v-(\alpha+1)i) \ln(A(T))}]}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v} = \frac{S_1(0) \phi^{\mathbb{P}^*}_{\ln A}(v-(\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}
\]
Characteristic function of $\ln(A(T))$ under $\mathbb{P}^*$:

- Under the measure $\mathbb{P}^*$ we have

$$
\phi_{\ln A}^{\mathbb{P}^*} = E^{\mathbb{P}^*}[e^{iu \ln(A(T))}]
= e^{iu(\ln(A(0))+(\omega_2-\omega_1+q_1-q_2)T)}E^{\mathbb{P}^*}[e^{iu((Y_2(T) - Y_1(T))+(a_2-a_1)Z(T))}]
= e^{iu(\ln(A(0))+(\omega_2-\omega_1+q_1-q_2)T)}E^{\mathbb{P}^*}[e^{iuX_A(T)}]
$$

where $X_A(T) = (Y_2(T) - Y_1(T)) + (a_2-a_1)Z(T)$

$$
\phi_{X_A}^{\mathbb{P}^*} = E^{\mathbb{P}^*}[e^{iuX_A(T)}] = E^{Q}[\frac{d\mathbb{P}^*}{dQ}e^{iu((Y_2(T) - Y_1(T))+(a_2-a_1)Z(T))}]
$$

$$
\frac{d\mathbb{P}^*}{dQ} = \frac{S_1(T)}{e^{rT}S_1(0)} = \frac{e^{Y_1(T)+a_1Z(T)}}{E^{Q}[e^{Y_1(T)+a_1Z(T)}]} = e^{-(\varphi Y_1(-i)T+\varphi Z(-a_1)T)}e^{Y_1(t)+a_1Z(t)}
$$

$$
\phi_{X_A}^{\mathbb{P}^*} = E^{Q}[e^{-(\varphi Y_1(-i)T+\varphi Z(-a_1)T)+(Y_1(T)+a_1Z(T))}e^{iu((Y_2(T) - Y_1(T))+(a_2-a_1)Z(T))}]
= e^{-\varphi Y_1(-i)T}E^{Q}[e^{i(-u-i)Y_1(T)}]E^{Q}[e^{iuY_2(T)}]e^{-\varphi Z(-ia_1)T}E^{Q}[e^{i(u(a_2-a_1)-ia_1)Z(T)}]
= e^{(-\varphi Y_1(-i)+\varphi Y_1(-u-i)+\varphi Y_2(u)-\varphi Z(-ia_1)+\varphi Z(u(a_2-a_1)-ia_1))T}
$$
Calibration

- Using the FFT method we can calibrate the $Y_1(t)$, $Y_2(t)$ and $Z(t)$ parameters and $a_1$ and $a_2$ with respect to the Vanilla market.

\[
\min_{a_1, Y_1, Z} \sum_{i,j} \left( \frac{C_{\text{mod}}(S_1, K^i, T^j) - C_{\text{mkt}}(S_1, K^i, T^j)}{C_{\text{mod}}(S_1, K^i, T^j) - C_{\text{mkt}}(S_1, K^i, T^j)} \right)^2
\]

\[
\min_{a_2, Y_2, Z} \sum_{i,j} \left( \frac{C_{\text{mod}}(S_2, K^i, T^j) - C_{\text{mkt}}(S_2, K^i, T^j)}{C_{\text{mod}}(S_2, K^i, T^j) - C_{\text{mkt}}(S_2, K^i, T^j)} \right)^2
\]

- And impose a correlation condition

\[
\min_{a_1, a_2, Y_1, Y_2, Z} \left( \text{Corr} \left( \ln(S_1(t)), \ln(S_2(t)) \right) - \text{Corr}^{\text{mkt}} \right)^2
\]

where the pairwise linear correlation coefficient is given by

\[
\text{Corr} \left( \ln(S_1(t)), \ln(S_2(t)) \right) = \frac{a_1 a_2 \text{Var}(Z(1))}{\sqrt{\text{Var}(Y_1(1)) + a_1^2 \text{Var}(Z(1))} \sqrt{\text{Var}(Y_2(1)) + a_2^2 \text{Var}(Z(1))}}
\]
Historical Correlation:
- The market correlation between asset log-returns is usually estimated on a time window of 125 days up to (and including) the valuation date.
  - No rigorous justification for that choice,
  - Sensitive to the length of the time window,
  - Especially if it includes crisis period or not.
- Historical correlation is a “real world” measure.
- Vanilla option calibration is done in a risk neutral world.
- In the Gaussian models the correlation is not affected by changes of measures but this is not the case once the assumption of Brownian motion as driving process is abandoned.

Implied Correlation:
- Fitting the implied correlation coming from the market of derivatives written on these two assets.
  1. We need market data.
  2. We need (Semi)-analytic pricing formula,
Quanto Futures and Option on the Nikkei 225

- Multi asset Options market data seem to be rare or even non-existent.
- However, there is a famous market of products written on two underlying assets:
  1. USD Nikkei 225 Index Quanto Futures,
  2. USD Nikkei 225 Index Quanto Options.

- These products are dependent upon two underlyings, namely,
  1. Nikkei 225 Index (JPY)
     - Price June 2014: $S(0) \approx 15000$ JPY
     - It is structured to reflect the Japanese stock market using the 225 top-rated Japanese companies, including such well-known firms as Honda, Canon and Sony.
  2. The USDJPY Foreign Exchange rate (Price today $X(0) \approx 102$ JPY/USD).

- These products are traded on the Chicago Mercantile Exchange, the largest and most diverse financial futures and options exchange in the world.
Index Futures

Index Futures contract:

- A legally binding agreement between two parties to pay or receive the difference between:
  1. the predicted underlying price set when entering into the contract
  2. the price of the underlying at maturity $S(T)$.

$$F(T) = E^{RN}[S(T)] = e^{rT}S(0)$$

Multiplier:

- Index futures are often traded with a multiplier that inflates the value of the contract to add leverage to the trade.

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Dow Jones</td>
<td>10</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>250</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>500</td>
</tr>
</tbody>
</table>

Example: if Dow Jones Index future price = 10,000 USD, this means that if an investor purchased one futures contract, it would be worth 100,000 USD.
Nikkei 225 Index Futures

- CME customers have a choice of currencies for this product:

  1. Yen-denominated CME Nikkei 225 futures (symbol: NIY)
     - Contract Multiplier: 500 JPY
     - Contract Months: Quarterlies and Serials
     - Minimum Price Change (Tick): 5 Index points

  2. Dollar-denominated CME Nikkei 225 futures (symbol: NKD)
     - Contract Multiplier: 5 USD
     - Contract Months: Quarterlies
     - Minimum Price Change (Tick): 5 Index points
Introduction : Nikkei 225 Index Futures

- Two CME Nikkei 225 Futures markets with Different Prices.
  1. Yen-denominated CME Nikkei 225 futures (May 2014) = 14185 (JPY)
  2. USD-denominated CME Nikkei 225 futures (May 2014) = 14200 (JPY)

- Contract value:
  1. JPY Nikkei 225 Index Futures (May 2014) : 14185 * 500 JPY
  2. USD Nikkei 225 Index Quanto Futures (May 2014) : 14200 * 5 USD

- Rem:
  1. JPY/USD FX spot rate ≈ 102 JPY/USD
  2. Nikkei 225 Index ≈ 14 100 JPY

- Two different worlds
  - JPY world (foreign) : \( F^f(T) = E^f[S(T)] = 14185 \)
  - USD world (domestic) : \( F^d(T) = E^d[S(T)] = 14200 \)
Contract Specifications: CME Nikkei 225 Futures

Please note that there are some important differences in contract specifications among the products in the CME Nikkei 225 futures complex.

<table>
<thead>
<tr>
<th>Contract Specifications</th>
<th>Dollar-Denominated</th>
<th>Yen-Denominated</th>
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<tbody>
<tr>
<td><strong>Ticker Symbols</strong></td>
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<td>Open Outry: Not applicable</td>
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<td>CME Globex Platform: NIK</td>
<td>CME Globex Platform: NY</td>
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<td><strong>Contract Size</strong></td>
<td>$ 5.00 x CME Nikkei 225 futures price</td>
<td>Y 500 x CME Nikkei 225 futures price</td>
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<tr>
<td><strong>Minimum Price</strong></td>
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<tr>
<td><strong>Fraction (Tick)</strong></td>
<td>Five index points = $25 dollars</td>
<td>Five index points = ¥ 2,500</td>
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<td><strong>Contract Months</strong></td>
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<td>Mar, Jun, Sep, Dec</td>
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<td><strong>CME Globex Trading Hours (Daylight Saving Time)</strong></td>
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<td>06:00 - 15:15</td>
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<td>Reopens 16:30 - 17:00</td>
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<tr>
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<td>Closes 17:00 - 18:00</td>
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<td></td>
<td>Reopens 15:30 - 16:30</td>
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<td>Closes 16:30</td>
<td>Closes 16:30</td>
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<td>(Does not reopen at 17:00)</td>
<td>(Does not reopen at 17:00)</td>
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<td><strong>Daily Settlement Time</strong></td>
<td>All trades executed after 15:15 settlement time will have the next day’s trade date</td>
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**Price Limits**

- **Lead Month Settlement**
  - 0 - 20,000: 1,000
  - 20,001 - 30,000: 1,500
  - 30,001 and up: 2,000

**Position Limits**

- 5,000 contracts

**Last Trading Day**

- Business day preceding the second Friday of the contract month immediately preceding the day of determination of the final settlement price

**Final Settlement Price**

- Based upon a Special Opening Quote of the Nikkei 225 Stock Average referencing the opening values of constituent stocks

*Applicable to both yen- and dollar-denominated contracts.*
### Nikkei/Yen Futures Quotes

#### Globex

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<th>Month</th>
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<th>Charts</th>
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<th>Change</th>
<th>Prior Settle</th>
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<th>Low</th>
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### Nikkei/USD Futures Quotes

**Globex**

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**Legend:**
- **OPT:** Options
- **Price Chart**
Nikkei 225 Index Futures

Black Scholes

- Nikkei 225 Index FRN dynamics (JPY): \(dS(t) = r_f S(t) dt + \sigma_S S(t) dB^{FRN}_S(t)\)
- JPY/USD FX rate dynamics FRN (JPY): \(dX(t) = (r_f - r_d) X(t) dt + \sigma_X X(t) dB^{FRN}_X(t)\)
  - \(r_f\) is the Japan risk free interest rate
  - \(r_d\) is the US risk free interest rate
  - \(\rho\) is the correlation between \(B^{FRN}_S\) and \(B^{FRN}_X\)

- Moving from the FRN to the DRN world:

\[
\eta_t = \left. \frac{d\mathbb{P}^d}{d\mathbb{P}^f} \right|_{\mathcal{F}_t} = \frac{e^{r_d t} X(t)}{e^{r_f t} X(0)}
\]

- Nikkei 225 Index dynamics DRN (USD): \(dS(t) = (r_f + \rho \sigma_S \sigma_X) S(t) dt + \sigma_S S(t) dB^{DRN}_S(t)\)

- JPY Nikkei 225 Index Futures (NIY) with maturity \(T\): \(F^f(T) = \mathbb{E}^f[S(T)] = e^{r_f T} S(0)\)
- USD Nikkei 225 Index Quanto Futures (NKD): \(F^d(T) = \mathbb{E}^d[S(T)] = e^{(r_f + \rho \sigma_S \sigma_X) T} S(0)\)
Nikkei 225 Index Futures

Introduction: Nikkei 225 Index Futures

- Spread:

\[ F_d(T) - F_f(T) = e^{r_f T} S(0) \left( e^{\rho \sigma X} - 1 \right) \]

- \( \rho > 0 \Rightarrow F_d(USD) > F_f(JPY) \), (Nikkei 225 index↑ JPY/USD FX spot ↑)
- \( \rho < 0 \Rightarrow F_d(USD) < F_f(JPY) \), (Nikkei 225 index↑ JPY/USD FX spot ↓)
- \( \rho = 0 \Rightarrow F_d = F_f \)

See also Jaeckel (2009).
Multi Lévy:

- Nikkei 225 index FRN dynamics (i.e. under $\mathbb{P}^f$):

$$S(t) = S(0)e^{\left(r_f + \omega_S\right)t + \gamma_S(t) + a_S Z(t)}$$

- USD-JPY FX spot rate (amount of JPY per unit of USD $X(0) = 102.48 JPY/USD$) FRN dynamics:

$$X(t) = X(0)e^{\left(r_f - r_d + \omega_X\right)t + \gamma_X(t) + a_X Z(t)}$$

where $\gamma_X(t)$, $\gamma_S(t)$ and $Z(t)$ are independent Lévy processes

- The common process $Z(t)$ can be considered as the systematic part of the risk
- The processes $\gamma_X(t)$ and $\gamma_S(t)$ can be seen as capturing the idiosyncratic shock

Risk Neutral adjustments:

- $\omega_S = -\varphi^f_{Y_S}(-i) - \varphi^f_{Z}(-a_S i)$.
- $\omega_X = -\varphi^f_{Y_X}(-i) - \varphi^f_{Z}(-a_X i)$. 
Nikkei 225 Index Futures

Multi Lévy

- JPY Nikkei 225 Index Futures (NIY) with maturity $T$:

$$F^f(T) = E^f[S(T)] = e^{rfT}S(0)$$

- USD Nikkei 225 Index Quanto Futures (NKD):

$$F^d(T) = E^d[S(T)] = E^f[\eta_T S(T)]$$

$$\eta_t = \left. \frac{d\mathbb{P}^d}{d\mathbb{P}^f} \right|_{\mathcal{F}_t} = \frac{e^{rd_t X(t)}}{e^{rf t X(0)}} = e^{-\varphi^f Y_X(-it - \varphi^f Z(-ia_X)t)} e^{Y_X(t) + a_X Z(t)}$$

This is in fact the density process of an Esscher change of measure, namely

$$\eta_t = e^{-\varphi L_X(-ih)t + hL_X(t)}$$


$$F^d(T) = S(0)e^{(rf - \varphi^f Z(-ia_iZ) - \varphi^f Z(-aS) + \varphi^f Z(-ia(aS + aX)))}T = S(0)e^{(rf + q)T}$$

- Adjustment $q$ only dependent on the $Z(t)$ parameters and $aS$ and $aX$.
JPY Nikkei 225 Index Futures (NIY) with maturity $T$:

$$F^f(T) = E^f[S(T)] = e^{rf}T S(0)$$

USD Nikkei 225 Index Quanto Futures (NKD):

$$F^d(T) = E^d[S(T)] = S(0)e^{(rf+q)T}$$

and therefore $q = \frac{1}{T} \ln \left( \frac{F^d(T)}{F^f(T)} \right)$

The quanto adjustment $q$ can be rewritten as

$$q = -\varphi_Z(-axi) - \varphi_Z(-asi) + \varphi_Z(-i(as + ax))T$$

$$= asax\sigma_Z^2 + \int_{\mathbb{R}} \left( e^{(as+ax)z} - e^{axz} - e^{asz} + 1 \right) \nu_Z(dz)$$

$$= \text{Cov} (L_S, L_X) + \sum_{n=3}^{\infty} \sum_{k=1}^{n-1} \frac{a_{S}^{n-k} a_{X}^{k}}{k!(n-k)!} \int_{\mathbb{R}} z^n \nu_Z(dz), \quad (2)$$

In the Brownian motion case, one find the well-known Black and Scholes quanto adjustment:

$$q = \text{Cov} (L_S, L_X) \quad (3)$$

In a Multi-Lévy framework, the quanto readjustment depends on higher order cumulants of the pure jump part of the systematic risk process.
VG case:

- Let $Z(t)$ be a $VG(\sigma_Z, k_Z, \theta_Z)$
- The characteristic function:

$$\phi^f_{Z_t}(u) = E^f[e^{iuZ(t)}] = \left(1 - iuk_Z\theta_Z + \frac{1}{2}k_Zu^2\sigma^2_Z\right)^{-\frac{t}{k_Z}}.$$

USD Nikkei 225 Index Quanto Futures (NKD):

$$F^d(T) = \frac{S(0)e^{rfT}\phi^f_{Z_T}(-i(as + ax))}{\phi^f_{Z_T}(-axi)\phi^f_{Z_T}(-asi)}$$

$$= S(0)e^{rfT} \left(\frac{1 - (as + ax)k_Z\theta_Z - \frac{1}{2}k_Z(as + ax)^2\sigma^2_Z}{(1 - axk_Z\theta_Z - \frac{1}{2}k_Za_x^2\sigma^2_Z) (1 - ask_Z\theta_Z - \frac{1}{2}k_Za_s^2\sigma^2_Z)}\right)^{-\frac{T}{k_Z}}$$

- quanto adjustment:

$$q = -\frac{1}{k_Z} \ln \left(\frac{1 - (as + ax)k_Z\theta_Z - \frac{1}{2}k_Z(a_s + ax)^2\sigma^2_Z}{(1 - axk_Z\theta_Z - \frac{1}{2}k_Za_x^2\sigma^2_Z) (1 - ask_Z\theta_Z - \frac{1}{2}k_Za_s^2\sigma^2_Z)}\right)$$
Calibration procedure:

- Step 1: The $Z$ parameters and the $a_S$ and $a_X$ coefficients can be calibrated with respect to the USD Nikkei 225 Index Quanto Futures market,

$$\min_{a_S,a_X,Z} \sum_i \left( F_{mod}^i(T) - F_{mkt}^i(T) \right)^2$$

- Step 2: Once the $Z$ parameters and the $a_S$ and $a_X$ coefficients are calibrated, we can efficiently obtain the $Y_S$, $Y_X$ parameters by calibration to Vanilla Options on individual stock $S$ (Nikkei 225) and $X$ (USDJPY FX spot rate),

$$\min_{Y_S} \sum_{i,j} \left( C_{mod}^{S}(S,K_{i},T_{j}) - C_{mkt}^{S}(S,K_{i},T_{j}) \right)^2$$

$$\min_{Y_X} \sum_{i,j} \left( C_{mod}^{X}(X,K_{i},T_{j}) - C_{mkt}^{X}(X,K_{i},T_{j}) \right)^2$$

- See e.g. Bossens et al. (2010) for a summary of the relevant conventions and the treatment of foreign-exchange market data.
Calibration procedure:

Because of the small number of Futures market data available, it is more efficient to calibrate everything in one step:

\[
\begin{align*}
\min_{a_S, a_X, Z} & \sum_i \left( F_{\text{mod}}(T^i) - F_{\text{mkt}}(T^i) \right)^2 \\
\min_{Y_S, a_S, Z} & \sum_{i,j} \left( C_{\text{mod}}(S, K^i, T^j) - C_{\text{mkt}}(S, K^i, T^j) \right)^2 \\
\min_{Y_X, a_X, Z} & \sum_{i,j} \left( C_{\text{mod}}(X, K^i, T^j) - C_{\text{mkt}}(X, K^i, T^j) \right)^2
\end{align*}
\]
FFT Method:

- The Carr-Madan type formula:

\[
C(K, T) = e^{-r_f T} E^f [(S(T) - K)^+] \\
= \frac{e^{-\alpha \ln(K)}}{\pi} \int_0^{+\infty} e^{-iv \ln(K)} \psi(v) dv
\]

where

\[
\psi(v) = \frac{e^{-r_f T} E^f [e^{i(v-(\alpha+1)i) \ln(S(T))}]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} = \frac{e^{-r_f T} \phi^f_\ln S(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}
\]

\[
\phi^f_\ln S(u) = E^f [e^{iu \ln(S(T))}]
\]
**Calibration : 13th June 2014**

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<th>USD Quanto Futures</th>
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<td>Sept (91d)</td>
<td>15030</td>
<td>15065, 15066,37</td>
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**USDJPY**

| θ_ν_X    | 0.1514      | θ_Z          | -0.1830     | θ_ν_S    | -0.0177 |
| σ_ν_X    | 0.0070      | σ_Z          | 0.1095      | σ_ν_S    | 0.0150  |
| k_ν_X    | 0.0449      | k_Z          | 0.0522      | k_ν_S    | 0.0084  |

| a_X      | 0.4008      | a_S          | 1.8110      |

ρ_{imp} 82%

**Nikkei 225**

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<th>mkt Mid imp vol</th>
<th>VIX vol</th>
<th>mkt Bid imp vol</th>
<th>mkt Ask imp vol</th>
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**Graphs**

- **USDJPY implied vol**
- **Nikkei 225 implied vol**
# USD Nikkei 225 Index Quanto Options

## Contract Specifications: CME Nikkei 225 Options on Futures

The contract specifications for options on dollar-denominated CME Nikkei 225 futures are as follows:

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<th></th>
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</thead>
<tbody>
<tr>
<td><strong>Ticker Symbols</strong></td>
<td>Calls: KN Puts: JN</td>
</tr>
<tr>
<td><strong>Underlying Contract</strong></td>
<td>One dollar-denominated CME Nikkei 225 futures contract</td>
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<tr>
<td><strong>Open Outcry Hours</strong></td>
<td>08:00 - 15:15</td>
</tr>
<tr>
<td><strong>Minimum Price</strong></td>
<td>Five index points ($25.00)</td>
</tr>
<tr>
<td><strong>Fluctuation (Tick)</strong></td>
<td>All 12 calendar months</td>
</tr>
<tr>
<td><strong>Last Day of Trading</strong></td>
<td>Quarterly options: Same date as underlying futures</td>
</tr>
<tr>
<td></td>
<td>Other eight months: Third Friday of the contract month</td>
</tr>
<tr>
<td><strong>Position Limits</strong></td>
<td>5,000 futures-equivalent contracts net on the same side of the market in all contract months combined</td>
</tr>
<tr>
<td><strong>Last Trading Day</strong></td>
<td>Business day preceding the second Friday of the contract month</td>
</tr>
<tr>
<td><strong>Final Settlement Date</strong></td>
<td>Based upon a Special Opening Quotation of the Nikkei 225 Stock Average Index referencing the opening values of constituent stocks</td>
</tr>
<tr>
<td><strong>Settlement Procedures</strong></td>
<td>Cash settlement to the Special Opening Quotation of the Nikkei Stock Average Index</td>
</tr>
<tr>
<td><strong>Option Exercise</strong></td>
<td>American Style. An option can be exercised until 7:00 p.m. Chicago time on any business day the option is traded. An option that is in-the-money and has not been exercised prior to the termination of trading shall, in the absence of contrary instructions, be delivered to CME Clearing by 7:00 p.m. on the day of determination of the Final Settlement Price, and be automatically exercised.</td>
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</table>
USD Nikkei 225 Index Quanto Options

- The arbitrage free price of a Quanto Call Option on the USD denominated Nikkei 225 Index (Quanto) Futures is given by

\[
QC(F^d(T_2), K, T_1) = e^{-r_d T_1} M_{\text{fix}} E^d [(F^d_{T_1}(T_2) - K)^+] \rightarrow \text{(USD)}
\]

1. \( M_{\text{fix}} = 5 \text{ USD} \) is the fixed Multiplier.
2. \( r_d \) is the US risk free interest rate.
3. \( F^d(T_2) \) is the USD Nikkei 225 Index Quanto Futures with maturity \( T_2 \) (symbol : NKD).
4. \( T_1 \leq T_2 \) is the maturity of the options

\[
F^d_{T_1}(T_2) = E^d [S(T_2)|\mathcal{F}_{T_1}] = E^f [\eta(T_2) S(T_2)|\mathcal{F}_{T_1}]
\]

\[
\eta_t = \left. \frac{d^d}{d^f} \right|_{\mathcal{F}_t} = \frac{e^{r_d t} X(t)}{e^{r_f t} X(0)} = e^{-\varphi^d Y_X(-i)t - \varphi^f(-a_X i) t} e^{Y_X(t)+a_X Z(t)}
\]
USD Nikkei 225 Index Quanto Options

- At time $T_1$ the USD Nikkei 225 Index Quanto Futures with maturity $T_2$ can be expressed in terms of the Nikkei 225 Index $S$ as following

$$F_{T_1}^d(T_2) = S(T_1)e^{(r_f+q)(T_2-T_1)}$$

- The arbitrage free price of a Quanto Call Option on the USD denominated Nikkei 225 Index (Quanto) Futures $QC(F^d(T_2), K, T_1)$ is given by

$$QC(F^d(T_2), K, T_1) = e^{-r_dT_1}M_{\text{fix}}E^d[(F_{T_1}^d(T_2) - K)^+]$$

$$= e^{-r_dT_1}M_{\text{fix}}Q_{adj}E^d[(S(T_1) - K^*)^+]$$

$$Q_{adj} = e^{(r_f+q)(T_2-T_1)} = e^{(r_f-\varphi_Z(-aX_1) - \varphi_Z(-aS_1) + \varphi_Z(-i(aS+aX)))(T_2-T_1)}$$

$$K^* = \frac{K}{Q_{adj}}$$
FFT Method:

- The Carr-Madan type formula:

\[
QC(F^d(T_2), K, T_1) = e^{-r_d T_1 M_{fix} Q_{adj} E^d[(S(T_1) - K^*)^+]} \\
= e^{-\alpha \ln(K^*)} \int_{0}^{+\infty} e^{-i\ln(K^*)} \psi(v) dv
\]

where

\[
\psi(v) = \frac{e^{-r_d T_1 M_{fix} Q_{adj} E^d[e^{i(v-(\alpha+1)i) \ln(S(T_1))}]}}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} = \frac{e^{-r_d T_1 M_{fix} Q_{adj} \phi_{ln S}^d(v - (\alpha + 1)i)}}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}
\]
Quanto Option

Characteristic function of $\ln(S(t))$ under $\mathbb{P}^d$ :

- Under the measure $\mathbb{P}^d$ we have

$$
\phi^d_{\ln S}(u) = E^d[e^{iu \ln(S(t))}]
= E^f[\eta_t e^{iu(\ln(S(0))+(r_f+\omega_S)t+(Y_S(t)+a_SZ(t))}]
= e^{iu(\ln(S(0))+(r_f+\omega_S)t)} e^{i\phi^f_{Y_S}(u)t} e^{(\varphi^f(u)+ia) \ln(S(0))+iaX)}

See also Eberlein et al. (2004) :

Lévy triplet of $Y_S(t)$ :

- Under $\mathbb{P}^f$ : $(b_{Y_S}, \sigma_{Y_S}^2, \nu_{Y_S}^f(dy))$
- Under $\mathbb{P}^d$ : $(b_{Y_S}, \sigma_{Y_S}^2, \nu_{Y_S}^f(dy))$

Lévy triplet of $Z(t)$ :

- Under $\mathbb{P}^f$ : $(b_Z, \sigma_Z^2, \nu_Z^f(dz))$
- Under $\mathbb{P}^d$ : $(b_Z + aX \sigma_Z^2 + \int_{\mathbb{R}} z(e^{axz} - 1) \nu_Z^d(dz), \sigma_Z^2, \nu_Z^d(dz) = e^{axz} \nu_Z^f(dz))$
VG case:

\[
\phi_{\ln S}(u) = e^{i u (\ln(S(0)) + (r_f + \omega_S) t)} \phi_Y^f(u) \phi_Z^f(ua_S - i\alpha)(\phi_Z(-i\alpha))^{-1}
\]

\[
\phi_Y^f(u) = \left(1 - i u k_Y \theta_Y + \frac{1}{2} k_Y u^2 \sigma_Y^2 \right)^{-\frac{1}{k_Y}}
\]

\[
\phi_Z^f(u) = \left(1 - i u k_Z \theta_Z + \frac{1}{2} k_Z u^2 \sigma_Z^2 \right)^{-\frac{1}{k_Z}}
\]

\[
\omega_S = -\varphi_Y^f(-i) - \varphi_Z^f(-a_S i)
\]

\[
= \frac{1}{k_Y} \ln \left(1 - k_Y \theta_Y - \frac{1}{2} k_Y \sigma_Y^2 \right) + \frac{1}{k_Z} \ln \left(1 - a_S k_Z \theta_Z - \frac{1}{2} k_Z a_S^2 \sigma_Z^2 \right).
\]
Model impact: Nikkei 225 (Quanto) Options

- **Quanto Call:**

\[
QC(S, K, T) = e^{-r_d T} E^d \left[ (S(T) - K)^+ \right] \\
= \frac{e^{-\alpha \ln(K)}}{\pi} \int_0^{+\infty} e^{-iv \ln(K)} \psi^d(v) dv
\]

where

\[
\psi^d(v) = \frac{e^{-r_d T} E^d \left[ e^{i(v-(\alpha+1)i) \ln(S(T_1))} \right]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}
\]

- **Call:**

\[
C(S, K, T) = e^{-r_f T} E^f \left[ (S(T) - K)^+ \right] \\
= \frac{e^{-\alpha \ln(K)}}{\pi} \int_0^{+\infty} e^{-iv \ln(K)} \psi^f(v) dv
\]

where

\[
\psi^f(v) = \frac{e^{-r_d T} E^f \left[ e^{i(v-(\alpha+1)i) \ln(S(T_1))} \right]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}
\]
### Quanto Call implied corr curve: 13th June 2014

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<th>$C_{mkt}$</th>
<th>$QC^{VG}$</th>
<th>$\rho_{SX}^{v1}$</th>
<th>$QC^{BS}(\rho_{SX}^{v1})$</th>
<th>$\rho_{SX}^{v2}$</th>
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<td>155,1</td>
<td>-100%</td>
<td>164,7</td>
<td>79%</td>
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<td>15625</td>
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<td>-100%</td>
<td>130,3</td>
<td>80%</td>
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### Hist corr Calibration: 13th June 2014

#### USDJPY

<table>
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<tr>
<td>$\theta_{Y_X}$</td>
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<tr>
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<td>$k_{Y_X}$</td>
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<td>$a_X$</td>
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#### Nikkei 225

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\[ \rho_{hist}^{128d} = 28\% \]
## Quanto Call hist corr impact: 13th June 2014

<table>
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<tr>
<th>$K$</th>
<th>$C^{mkt}$</th>
<th>$QC^{VG}$</th>
<th>$QC^{VG}(\rho_h)$</th>
<th>$QC^{BS}_{V1}(\rho_h)$</th>
<th>$QC^{BS}_{V2}(\rho_h)$</th>
<th>$QC^{BS}(\rho_{imp})$</th>
<th>$QC^{BS}(\rho_{imp})$</th>
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<td>315,3</td>
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<td>168,9</td>
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<td>133,8</td>
<td>117,8</td>
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<td>119,6</td>
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</table>

### Quanto Call implied correlation

![Quanto Call implied correlation graph](image-url)
## Time Evolution

<table>
<thead>
<tr>
<th></th>
<th>Friday 13/06/14</th>
<th>Monday 16/06/14</th>
<th>Tuesday 17/06/14</th>
<th>Wednesday 18/06/14</th>
<th>Thursday 19/06/14</th>
<th>Friday 20/06/14</th>
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<tr>
<td>$F^f$</td>
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<td>14950</td>
<td>15030</td>
<td>15100</td>
<td>15365</td>
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<td>15065</td>
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<td>15060</td>
<td>15130</td>
<td>15390</td>
<td>15490</td>
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<td>$F^d_{VG}$</td>
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<td>14986,64</td>
<td>15059,95</td>
<td>15131,10</td>
<td>15388,83</td>
<td>15488,31</td>
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<tr>
<td>$T_{(days)}$</td>
<td>91</td>
<td>88</td>
<td>87</td>
<td>86</td>
<td>85</td>
<td>84</td>
</tr>
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<td>$\sigma^S_{ATM}$</td>
<td>19,56%</td>
<td>18,13%</td>
<td>18,70%</td>
<td>17,04%</td>
<td>15,63%</td>
<td>18,34%</td>
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<tr>
<td>$\sigma^X_{ATM}$</td>
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<td>5,51%</td>
<td>5,42%</td>
<td>5,55%</td>
<td>4,98%</td>
<td>4,87%</td>
</tr>
<tr>
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<td>83%</td>
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<td>90%</td>
<td>94%</td>
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<td>$\rho_{imp}^{VG}$</td>
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<td>94%</td>
<td>74%</td>
<td>77%</td>
<td>78%</td>
<td>85%</td>
</tr>
<tr>
<td>$\rho_{hist}$ (128d)</td>
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<td>28,78%</td>
<td>29,29%</td>
<td>30,45%</td>
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<td>0,00873</td>
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<td>Cov$_{VG}$ ($L_S, L_X$)</td>
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<td>-0,00025</td>
<td>0,00005</td>
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<td>-0,00011</td>
</tr>
</tbody>
</table>
## Time Evolution

<table>
<thead>
<tr>
<th></th>
<th>Friday 13/06/14</th>
<th>Monday 16/06/14</th>
<th>Tuesday 17/06/14</th>
<th>Wednesday 18/06/14</th>
<th>Thursday 19/06/14</th>
<th>Friday 20/06/14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{1y}^{hist}$</td>
<td>37.70%</td>
<td>39.39%</td>
<td>39.43%</td>
<td>39.58%</td>
<td>38.88%</td>
<td>40.04%</td>
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<tr>
<td>$\rho_{5m}^{hist}$</td>
<td>29.39%</td>
<td>29.20%</td>
<td>28.51%</td>
<td>28.12%</td>
<td>27.78%</td>
<td>29.99%</td>
</tr>
<tr>
<td>$\rho_{3m}^{hist}$</td>
<td>33.65%</td>
<td>34.35%</td>
<td>35.30%</td>
<td>33.31%</td>
<td>34.34%</td>
<td>29.99%</td>
</tr>
<tr>
<td>$\rho_{1m}^{hist}$</td>
<td>43.15%</td>
<td>49.78%</td>
<td>50.93%</td>
<td>48.29%</td>
<td>43.67%</td>
<td>43.68%</td>
</tr>
</tbody>
</table>

Griselda Deelstra (ULB)

grdeelst@ulb.ac.be

Strasbourg, Sept 18 2014
For $l_S, l_X \downarrow -\infty$, $\mathbb{P}(L_S(t) < l_S, L_X(t) < l_X) > 0$ if and only if $\text{corr}(L_X(t), L_S(t)) > 0$ for all $t > 0$.

For $l \downarrow -\infty$, $\mathbb{P}(L_S(t) < l, L_X(t) < l) > 0$:
Upper Tail dependence

For $l_S, l_X \uparrow \infty$, $\mathbb{P}(L_S(t) > l_S, L_X(t) > l_X) > 0$ if and only if $\text{corr}(L_X(t), L_S(t)) > 0$ for all $t > 0$.

For $l \uparrow \infty$, $\mathbb{P}(L_S(t) > l, L_X(t) > l) > 0$:
Conclusions:

- Calculations in this model are easy by the characteristic functions (independence between the components); by propositions about the characteristic exponents under an Esscher change of measure and by the well-known FFT method which meanly depends upon the characteristic function (under the right probability measure).
- The model is very flexible and can adjust any market values. We did a test and all correlations are reachable.
- The quanto adjustment in the formula of Quanto Futures equals a covariance term and a jump term;
- If the correlation is positive (and the jump term has not too big influence) then the foreign denominated futures have a lower price than the domestic denominated futures.
- Implied correlation can be much different from the historical correlation, and can also behave differently in the time evolution.
- For option pricing, using the implied volatility smile of the underlying is very important. When using this smile (instead of assuming a constant at the money volatility) implies that using the historical correlation instead of the implied correlation does not lead to big errors.
- This model with Variance Gamma processes has non-zero tail dependence, unlike the Gaussian model.

Future Research:

- Modelling subportfolios by such a model and calculate the influence of the dependence upon the solvency requirements


Guillaume Florence. Sato two-factor models for multivariate option pricing The journal of computational finance, 159-192 (2012)


Part 2 : Modelling correlated processes by Wishart processes

Joint work with Martino Grasselli (Università di Padova and De Vinci Finance Lab, Paris la Defense) and Christopher Van Weverberg (ULB)
The Wishart process is a matrix variate extension of the Cox-Ingersoll-Ross process, i.e. a solution of a one-dimensional stochastic differential equation of the form

\[ dS_t = \sigma \sqrt{S_t} dB_t + a(b - S_t)dt, \quad S_0 = s_0 \in \mathbb{R}^+ \]

with positive numbers \( a, b, \sigma \) and a one-dimensional Brownian motion \((B_t)_{t \geq 0}\):

- \( b \) long term rate
- \( a \) mean reversion speed
- \( \sigma \) the volatility
- \( 2ab \geq \sigma^2 \) for strict positive interest rates.
The Wishart process is a matrix variate extension of the Heston (1993) model, i.e. a solution of the stochastic differential equation of the form

\[
dS_t = r_t S_t \, dt + \sqrt{v_t} S_t \, dW^1_t \\
dv_t = \kappa (\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dW^2_t \\
dW^1_t \, dW^2_t = \rho \, dt
\]  

(4)

with
- \( \theta \) long term variance
- \( \kappa \) mean reversion speed
- \( \sigma \) the vol-of-vol
- \( \rho \) correlation between stock and volatility
- \( 2\kappa \theta \geq \sigma^2 \) for strict positive variances.
Why extending the Heston model?


- The dynamics of the implied volatility surface is driven by several factors.
- On the FX market the skew is stochastic.
- There exists a term structure of skew: short term skew ≠ long term skew.
- Calibration of the Heston model leads to excessive values for $\rho$. 
The Wishart process, introduced by Bru (1991), is defined by a solution of the $d \times d$-dimensional stochastic differential equation of the form

$$dS_t = \sqrt{S_t} dB_t Q + Q^T dB_t^T \sqrt{S_t} + (S_t K + K^T S_t + \alpha Q^T Q) dt$$

where $Q$ and $K$ are real valued $d \times d$-matrices, $\alpha$ a non-negative number, $(B_t)_{t \geq 0}$ a $d \times d$-dimensional Brownian motion and $\mathcal{M}_d(\mathbb{R})$ the set of all $d \times d$ matrices with entries in $\mathbb{R}$.

- $S_t$ symmetric matrix $(d \times d)$ square root process
- $\alpha \geq d + 1$ to guarantee strong existence of unique solutions; $\alpha \geq d - 1$ to guarantee weak existence.
- $Q$ the volatility
- $K$ models mean reverting behavior when negative definite
Motivation

The Wishart processes belong to the class of affine processes on positive semidefinite matrices characterized by Cuchiero and al. (2011), namely time homogeneous and stochastically continuous Markov processes for which the Laplace transform has exponential-affine dependence on the initial state:

$$\mathbb{E}_x \left[ e^{-\langle u, X_t \rangle} \right] = e^{-\phi(t,u) - \langle \psi(t,u), x \rangle},$$

for all $t$ and $u, x \in S^+_d$, for $\phi : \mathbb{R}_+ \times S^+_d \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S^+_d \to S^+_d$ with $S^+_d$ the set of all symmetric positive semidefinite matrices of $\mathcal{M}_d(\mathbb{R})$. 
Financial applications

Wishart processes have several interests:

- A generalization of stochastic volatility in multidimension models
- Defining the natural correlation between processes

It allows for semi-closed form solutions for a number of financial problems such as:

- Term structure of T-bonds and corporate bonds
- Option pricing in multivariate stochastic volatility models
- Structural model for credit risk.
Wishart processes have been recently applied in finance by Gouriéroux and Sufana (2004). They model the dynamics of $p$ risky assets by:

$$dS_t = \text{diag}(S_t)[(r1 + \lambda_t)dt + \sqrt{X_t}dZ_t]$$

where $r$ is a positive number, $1 = (1, \ldots, 1)^\top \in \mathbb{R}^p$, $\lambda_t$ is a $p$-dimensional stochastic process interpreted as the risk premium, and $Z_t$ a $p$-dimensional Brownian motion. The volatility process $X_t$ is a Wishart process.

Gouriéroux and Sufana (2004) suppose the volatility matrix to be independent of $Z_t$. Da Fonseca and al. (2008) relax the independence assumption and assume

$$Z_t = B_t\sqrt{1 - [RR^\top]} + W_tR^\top$$

with $R$ a constant $p$ dimensional matrix which describes the correlation structure. They derive e.g. in the case when $p = 1$ (so one single asset), the stochastic correlation between the stock noise and the noise of the volatility of the stock:

$$\text{Corr}_t(\text{Noise}(S), \text{Noise}(\text{Vol}(S))) = \frac{\text{Tr}[RQX_t]}{\sqrt{\text{Tr}[X_t]}\sqrt{\text{Tr}[Q^\top QX_t]}}$$
Da Fonseca and al. (2008) relax the model further but concentrate upon $p$ risky assets. They have chosen the adequate correlation between the noises driving the asset and the noises driving the volatility process in order to take into account the skew effect of the volatility smile, and to keep the model affine.

\[
dZ^k_t = \sqrt{1 - \text{Tr}[R_k R_k^\top]} dB^k_t + \text{Tr}[R_k dW_t^\top], \quad k = 1, \ldots, n
\]

where

\[
R_k = \begin{pmatrix}
0 & 0 & 0 \\
\rho_1 & \ldots & \rho_n \\
0 & 0 & 0
\end{pmatrix},
\]

where $\rho_i \in [-1, 1]$, for $i = 1, \ldots, n$ and $\rho^\top \rho \leq 1$. 
Stylized effects

As shown in Lewis (2000), in the Heston model the main contribution to the short term skew is given by the volatility of volatility multiplied by the (scalar) correlation between the asset noise and the volatility noise.

In the case of 2 assets $S_1$ and $S_2$ with log returns $Y^1$ and $Y^2$ this correlation becomes in the Wishart stochastic volatility model of Da Fonseca and al. (2008)

$$
Corr_t \left( \text{Noise}(Y^1), \text{Noise}(\text{Vol}(S^1)) \right) = \frac{\langle Y^1, \Sigma^{11} \rangle_t}{\sqrt{\Sigma_t^{11}} \sqrt{\langle \Sigma^{11} \rangle_t}} = \frac{Q_{11} \rho_1 + Q_{21} \rho_2}{\sqrt{Q_{11}^2 + Q_{21}^2}}
$$

and in the Heston model (1993)

$$
Corr_t \left( \text{Noise}(Y^1), \text{Noise}(\text{Vol}(S^1)) \right) = \rho_1 \\
Corr_t \left( \text{Noise}(Y^2), \text{Noise}(\text{Vol}(S^2)) \right) = \rho_2
$$

Hence, in the Wishart model, the correlation between returns and volatilities of each asset involves all the coefficients $\rho_i$ through the off-diagonal parameters of the matrix $Q$. 
Numerical illustration

In order to visualize the above effects, we consider the next example from Da Fonseca et al. (2008):

\[
K = \begin{pmatrix} -2.5 & -1.5 \\ -1.5 & -2.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.21 & -0.14 \\ 0.14 & 0.21 \end{pmatrix}
\]

\[
\rho_1 = \rho_2 = -0.6.
\]

The Gindikin coefficient is \( \alpha = 7.14286 \), the interest rate \( r = 0 \) and the initial volatility matrix is given by

\[
\Sigma_0 = \begin{pmatrix} 0.09 & -0.036 \\ -0.036 & 0.09 \end{pmatrix}
\]

Observe that the dynamics of the two assets differ only for the off-diagonal term in \( Q \) while \( \sqrt{Q_{12}^2 + Q_{22}^2} = \sqrt{Q_{11}^2 + Q_{21}^2} \) are kept constant. With this choice the differences between the implied volatilities of asset 1 and asset 2 quantify the impact that the off-diagonal terms in the matrix \( Q \) have on the implied volatility surfaces, which are shown in the following figures.
Figure 1 and 2 are taken from Da Fonseca et al. (2008):

**Figure:** 1. Implied volatility surface on first asset

**Figure:** 2. Implied volatility surface on second asset

These figures show the typical smile-skew effect consistent with the Heston model. Since the correlations between returns and volatility are given by

\[
\text{Corr}_t \left( \text{Noise}(Y^1), \text{Noise}(\text{Vol}(Y^1)) \right) = \frac{Q_{11}\rho_1 + Q_{21}\rho_2}{\sqrt{Q_{11}^2 + Q_{21}^2}} = -0.832
\]

\[
\text{Corr}_t \left( \text{Noise}(Y^2), \text{Noise}(\text{Vol}(Y^2)) \right) = \frac{Q_{12}\rho_1 + Q_{22}\rho_2}{\sqrt{Q_{12}^2 + Q_{22}^2}} = -0.166.
\]

In line with the Lewis analysis, a higher (absolute) level of the correlation implies a more pronounced skew effect of the first asset with respect to the second one.
Future Research:

- Modelling subportfolios by such a model and calculate the influence of the dependence upon the solvency requirements
- Dynamic dependence between interest rate processes, market prices (stock, real estate), longevity and inflation

We first concentrated upon the maximal domain of Wishart Laplace transforms in

Let \((S_t)_{t \geq 0}\) be a price process.

**Definition (Moment explosion)**

Let \(u \in \mathbb{R}\). A moment explosion occurs at time \(\tau(u)\), if

\[
\tau(u) := \sup \{ t \geq 0 : \mathbb{E}[S_t^u] < \infty \} < \infty.
\]

and say that no moment explosion occurs if \(\tau(u) = \infty\). \(\tau(u)\) is called the moment explosion time.

Lee’s moment formula

**Theorem**

Let

\[ p^* := \sup \{ p : \mathbb{E}[S_T^{1+p}] < \infty \} \quad \text{and} \quad q^* := \sup \{ q : \mathbb{E}[S_T^{-q}] < \infty \} \]

and

\[ \beta_R := \limsup_{x \to +\infty} \frac{l^2(x)}{|x|/T} \quad \text{and} \quad \beta_L := \limsup_{x \to -\infty} \frac{l^2(x)}{|x|/T} \]

where \( x \) is the (log)-moneyness and \( l(x) \) the Black-Scholes implied volatility at moneyness \( x \). Then \( \beta_L, \beta_R \in [0, 2] \) and

\[ \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2} = p^* \quad \text{and} \quad \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2} = q^* \]

where \( \frac{1}{0} = \infty \).
Financial applications

1. Skew extrapolation: when using splines to parametrically extrapolate volatility skews beyond the actively traded strikes, the moment formulas raise warnings against spline functions that grow faster than $|x|^{1/2}$ and slower than $|x|^{1/2}$.

2. Model calibration: in each moment formula, the left-hand side is, in principle, observable from options data while the right-hand side is, in a wide class of models, easily calculated from parameters. By building this direct link between data and parameters, the moment formula can facilitate the calibration procedure by giving restrictions on the model’s parameters and therefore the moment formula can provide initial guesses for some or all of the parameters.

From an actuarial point of view, these kind of results are also interesting since life insurance companies and pension funds often issue policies/contracts with embedded options.
Let \((X_t)_{t \geq 0}\) be a Wishart process. We consider the following dynamics for \((X_t)_{t \geq 0}\)

\[
dX_t = \delta \Sigma^2 dt + \sqrt{X_t} dW_t \Sigma + \Sigma dW_t^T \sqrt{X_t}, \quad X_0 = x.
\]

with \(\delta \geq d - 1, \Sigma \in S^+_d\). We will denote such a process \(\mathcal{W}(0, \Sigma, \delta, x)\).

We denote by \(\Psi_{u,\mu}\) the joint Laplace transform of \(X\) :

\[
\Psi_{u,\mu}(t) = \mathbb{E} \left[ e^{-Tr(uX_t) - Tr(\int_0^t \mu X_s ds)} \right]
\]

where \(Tr\) denotes the trace operator and \(u, \mu\) are symmetric matrices for which this expression is finite.

The explicit joint Laplace transform

Proposition

Let \((X_t)_{t \geq 0}\) be a Wishart process \(\mathcal{W}(0, \Sigma, \delta, x)\). We have the following expressions for the joint Laplace transform:

\[
\Psi_{u, \mu}(t) = \mathbb{E}_x \left[ e^{-\text{Tr}(uX_t) - \text{Tr}\left( \int_0^t \mu X_s ds \right)} \right] = (\det(F_t(u, \mu)))^{-\frac{\delta}{2}} \exp \left( -\text{Tr}(F_t(u, \mu))^{-1} G_t(u, \mu)x \right)
\]

where

1. If \(\mu\) is positive semidefinite:
   \[
   F_t(u, \mu) = \cosh(t \sqrt{2\mu \Sigma^2}) + 2u \Sigma^2 \sinh(t \sqrt{2\mu \Sigma^2})(2\mu \Sigma^2)^{-1/2}
   \]
   \[
   G_t(u, \mu) = \sinh(t \sqrt{2\mu \Sigma^2})(2\mu \Sigma^2)^{-1/2} \mu + u \cosh(t \sqrt{2\Sigma^2 \mu})
   \]

2. If \(\mu\) is negative semidefinite:
   \[
   F_t(u, \mu) = \cos(t \sqrt{-2\mu \Sigma^2}) + 2u \Sigma^2 \sin(t \sqrt{-2\mu \Sigma^2})(-2\mu \Sigma^2)^{-1/2}
   \]
   \[
   G_t(u, \mu) = \sin(t \sqrt{-2\mu \Sigma^2})(-2\mu \Sigma^2)^{-1/2} \mu + u \cos(t \sqrt{-2\Sigma^2 \mu})
   \]
Theorem

Let \((X_t)_{t \geq 0}\) be a Wishart process \(\mathcal{W}(0, \Sigma, \delta, x)\) and

\[
\Psi_{u, \mu}(t) = \mathbb{E}\left[e^{-Tr(uX_t)} - Tr\left(\int_0^t \mu X_s ds\right)\right]
\]

his joint Laplace transform. Under the assumption that

\[
u \Sigma^2 \mu = \mu \Sigma^2 u
\]

then if \(u\) is not positive semidefinite, the explosion time \(T_{\text{exp}}\) of \(\Psi_{u, \mu}\) is given by

\[
T_{\text{exp}} = \inf_{i \in \mathcal{I}} \left\{ \frac{1}{\sqrt{\lambda_i}} \arctanh \left( \frac{-\sqrt{\lambda_i}}{\beta_i} \right) \mathbb{I}\{\lambda_i > 0\} + \frac{1}{\sqrt{-\lambda_i}} \arctan \left( \frac{-\sqrt{-\lambda_i}}{\beta_i} \right) \mathbb{I}\{\lambda_i < 0\} \right\}
\]

where \(\mathcal{I} = \{1, 2, \ldots, n\}\) and \(\lambda_i\) (resp. \(\beta_i\)) denotes the eigenvalue of \(2\mu \Sigma^2\) (resp. \(2u \Sigma^2\)).

If \(u\) is positive semidefinite then there is no explosion time.
The Integrated Laplace transform

Now we consider $\Psi_\mu$ the integrated Laplace transform of $X$:

$$\Psi_\mu(t) = \mathbb{E} \left[ e^{-Tr(\int_0^t \mu X_s ds)} \right] \quad \mu \in S_d$$

where $S_d$ is the set of symmetric matrices.

We deduce the explosion time for $\Psi_\mu$ as a corollary of the previous theorem.

**Corollary**

Let $(X_t)_{t \geq 0}$ be a Wishart process $\mathcal{W}(0, \Sigma, \delta, x)$ and

$$\Psi_\mu(t) = \mathbb{E} \left[ e^{-Tr(\int_0^t \mu X_s ds)} \right] \quad \mu \in S_d$$

his integrated Laplace transform. If $\mu$ is not positive semidefinite then the explosion time $T_{\text{exp}}$ of $\Psi_\mu$ is given by

$$T_{\text{exp}} = \frac{\pi}{2\sqrt{-\gamma_{\text{min}}}}$$

where $\gamma_{\text{min}}$ denotes the minimum (negative) eigenvalue of $2\mu \Sigma^2$. 
Ostrowski’s theorem

The corollary gives the explosion time in terms of the properties of the matrix $2\mu \Sigma^2$. Our goal is to give some bounds for the explosion time in terms of the eigenvalues of the matrix $\mu$.

The following theorem is due to Ostrowski (1959).

**Theorem**

Given two Hermitian matrices $A, B$ that are congruent each other through $A = SBS^T$ ($S$ invertible), with eigenvalues resp. $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_d$ and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_d$; if $\lambda_k \neq 0$ it follows

$$\sigma_{\text{max}} \geq \frac{\mu_k}{\lambda_k} \geq \sigma_{\text{min}},$$

where $\sigma_{\text{max}}$ (resp. $\sigma_{\text{min}}$) is the maximum (resp. the minimum) eigenvalue of the (positive definite) matrix $SS^T$. 
The next theorem gives some bounds for the explosion time in terms of the eigenvalues of the matrix $\mu$.

**Theorem**

The explosion time is finite if and only if the matrix $\mu$ is not positive semidefinite (that is $\mu$ has at least a negative eigenvalue). In this case, the explosion time $T_{\text{exp}}$ belongs to the following interval:

$$\frac{\pi}{2\sqrt{-2\lambda_{\text{min}}\sigma_{\text{max}}}} \leq T_{\text{exp}} \leq \frac{\pi}{2\sqrt{-2\lambda_{\text{min}}\sigma_{\text{min}}}}$$

where $\lambda_{\text{min}}$ denotes the minimum (negative) eigenvalue of $\mu$ and where $\sigma_{\text{max}}$ (resp. $\sigma_{\text{min}}$) is the maximum (resp. minimum) eigenvalue of the (positive definite) matrix $\Sigma^2$. 
The Laplace transform

Now we consider $\Psi_u$ the Laplace transform of $X$.

**Corollary**

Let $(X_t)_{t \geq 0}$ be a Wishart process $\mathcal{W}(0, \Sigma, \delta, x)$ and

$$
\Psi_u(t) = \mathbb{E}\left[ e^{-Tr(uX_t)} \right] \quad u \in S_d
$$

his Laplace transform. If $u$ is not positive semidefinite then the explosion time $T_{\text{exp}}$ of $\Psi_u$ is given by

$$
T_{\text{exp}} = -\frac{1}{\beta_{\text{min}}}
$$

where $\beta_{\text{min}}$ denotes the minimum (negative) eigenvalue of $2u\Sigma^2$. 
As in the case of the Integrated Laplace transform, we can give some bounds for the explosion time in terms of the eigenvalues of the matrix $u$.

**Theorem**

The explosion time is finite if and only if the matrix $u$ is not positive semidefinite. In this case, the explosion time $T_{\text{exp}}$ belongs to the following interval:

$$\frac{-1}{2\lambda_{\text{min}}\sigma_{\text{max}}} \leq T_{\text{exp}} \leq \frac{-1}{2\lambda_{\text{min}}\sigma_{\text{min}}}$$

where $\lambda_{\text{min}}$ denotes the minimum (negative) eigenvalue of $u$ and where $\sigma_{\text{max}}$ (resp. $\sigma_{\text{min}}$) is the maximum (resp. the minimum) eigenvalue of the (positive definite) matrix $\Sigma^2$. 
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Thank you for your attention