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VISUALIZATION AT UNIVERSITY LEVEL. THE CONCEPT OF INTEGRAL

Abstract. In recent years, several studies have highlighted the importance of tackling the students' difficulties in understanding of the concept of the integral. This study carried out with the first year students of the Mathematics Degree at the Universidad Complutense de Madrid, presents a deeper insight into these difficulties through data collected from a non-routine problem questionnaire and semi-structured interviews. Some of these difficulties clearly have their origin in the coordination between the analytic and graphic registers. In the analysis of students' use of the graphic register, the distinction between two different functions of images (iconic and heuristic) is exploited productively. Moreover, a specific teaching of visualization is recommended. As a main contribution in this approach two examples of relevant characteristic of visualization that should be taken into account in this proposal are shown: a high cognitive requirement and the need of a global apprehension.

Résumé. Visualisation au niveau universitaire : le concept d'intégrale. Ces dernières années, plusieurs études ont relevé l'importance d'aborder les difficultés des étudiants dans la compréhension de la notion d'intégrale. L'étude présentée, menée avec des étudiants de première année de la Licence de mathématiques à l'Université Complutense de Madrid, présente une vue plus approfondie de ces difficultés par les données recueillies à partir d'un questionnaire de problèmes non routiniers et d'entretiens semi-structurés. Certaines de ces difficultés ont évidemment leur origine dans la coordination entre les registres graphique et algébrique. Dans l'analyse de l'usage par les étudiants du registre graphique, la distinction entre deux fonctions différentes des images (iconique et heuristique) apparaît fructueuse. Par ailleurs, un enseignement spécifique de la visualisation est recommandé. A titre de principale contribution dans cette direction, on montre deux exemples de caractéristiques pertinentes de la visualisation, qui devraient être prises en compte dans cette proposition : une forte exigence cognitive et la nécessité d'une appréhension globale.

Mots clés. Intégrale, Enseignement de l'analyse, Représentations, Visualisation, Pensée mathématique avancée.

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1. Introduction

The results shown in this paper are part of a larger study (Souto, 2009; Souto & Gómez-Chacón, 2009) developed in the academic year 2008/2009. The main objective of this study was to improve the teaching of Mathematical Analysis at the university level favouring the processes of visualization.

This paper focuses on the concept of integral because it is one of the most prominent in the research on students' difficulties in Mathematical Analysis (Mundy, 1987; Llorens & Santoja, 1997; González-Martín & Camacho, 2005;

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Legrand, 2002). These studies emphasize that during the first year of university, students use the concept of integral in a very mechanical and operative way. This may be due to a wide gap between the teaching of the integral in high school and its instruction at the university level. However, in this paper the focus is not in institutional or sociocultural aspects of the understanding of the integral, but in cognitive ones. In this way, we focus on previous research results about the lack of integration between the concept of area and the integral.

This result leads us to pay particular attention to the coordination of graphic and analytic registers in an attempt to improve comprehension of the concept and eventually, to take into consideration the cognitive theories of the registers of semiotic representation (Duval, 1995, 1999, 2006; Hitt, 2003, 2006) in order to explain and analyze students' difficulties with the concept of integral. Furthermore, this study will go beyond mere identification and explanation of these said difficulties by exploring some recommendations for the teaching of the concept. One noteworthy recommendation is to pay explicit attention to visualization (Guzmán, 1996; Llorens & Santoja, 1997; González-Martín & Camacho, 2005) which will be elaborated further in this paper. According to Guzmán (1996, p. 40):

> *We should try to explicitly teach to perform the processes of visualization correctly. We should pay special attention to the different types of visualization and to their specific usefulness in the mathematical teaching and learning. We should try to be aware of the process of codification and decodification implied in the practice of visualization and try to make them explicit for our students.*

From the point of view of research on visualization, this study follows some of the approaches in which deeper research is needed pointed out by Presmeg (2006). Moreover, in this review (Presmeg, 2006) of the last thirty years of research on visualization in the field of Mathematics Education, lack of studies of visualization at University Level is highlighted. Our research tries to contribute in this way.

Therefore, this research poses the following questions:

- What kind of understanding of the concept of the integral do university level students possess and exhibit?
- What are the possibilities that visualization offers to improve the students' learning of the concept of the integral?

The above queries are treated in an exploratory way through a dialogue between theoretical and practical elements. Review of the theories serves as the basis of the analyses of the students' answers which will be put forward to promote examination of two main issues: (1) the identification of aspects related to the students' understanding of the concept of integral paying special attention to students' use of the graphic register (difficulties, mental blocks and misinterpretations as well as representations, treatments and conversions that appear in the students' answers) and (2) the exploration of characteristics of visualization to be taken into account in the design of a teaching proposal that improves students' understanding of the concept (the high cognitive requirement of visualization and the need for a global apprehension of the images). Moreover, a contribution of this paper is the use of distinction between *iconic* and *heuristic function of images* (Duval, 1999) to analyze students' productions. In this way, the *heuristic function* is found related to *visual methods* (Presmeg, 1985) and this connection inspires us to distinguish three different kinds of methods for solving a problem using the graphic register: *non-visual, mixed* and *visual*.

2. Theoretical Foundations

On one hand, the approach to the problem leads us to consider the theoretical framework of the cognitive theories of the registers of semiotic representation (Duval, 1995, 1999, 2006; Hitt, 2003, 2006) which talks about the coordination between registers and in particular, about the use of graphic register. This framework is useful in describing and analyzing students' difficulties in learning the concept of the integral.

On the other hand, research in Mathematics Education on visualization incorporates other perspectives of analysis that go beyond the semiotic approach which are relevant to our study. Among them, we highlight the role of visualization related to intuition in mathematical reasoning (De Guzmán, 1996; Arcavi, 2003), individual differences in visualization preference (Presmeg, 1985, 1991, 1994, 1995, 2006); and causes for the reluctance to visualization (Eisenberg & Dreyfus, 1991) that include more sociocultural issues about the use of visualization.

Understanding and learning of the mathematical concepts

We agree with Duval (1995, 1999) that the only possible access to mathematical objects is through their representations in their different semiotic registers. This fact makes the study of representations so important in order to explain the understanding of the concepts and the learning of mathematics.

From this point of view, the *understanding of a concept* is built through tasks that imply the use of different representation systems and promote the flexible articulation between representations. This articulation is produced through three cognitive activities inherent to any representation: representation, treatment and conversion. Thus, *learning mathematics* implies "*the construction of a cognitive structure by which the students can recognize the same object through different representations and can make objective connections between deductive and empirical mathematics*" (Duval, 1999, p. 12).

In this context, to *improve learning* supposes a diminution of the difficulties, misinterpretations and mental blocks that could surge in the mentioned process of articulation. As an example of these difficulties, observations highlight that "if most of the students can learn some *treatment*, very few of them can really *convert* representations" (Duval, 1999, p. 9). As we remarked in the introduction, research shows that the main cause of this is the lack of coordination between graphic and analytic registers. In our specific case of understanding the concept of integral, the analytic register in the students' answers is more prevalent than the graphic one. This fact conducts us to pay special attention to the use of the graphical register.

We define the notion of *visual comprehension* as the process in which a subject *acquires representations* of this concept in the *graphic register* and is able to *treat* and *convert* them into another register when thinking mathematically as well. This definition is based on a more global notion of understanding defined previously though it is limited to the graphic register. This limitation offers a new shade of meaning crucial to the study of the visualization processes.

Visualization in Mathematics Education

In fact, the specific teaching of visualization is proposed as a fundamental way in acquiring visual comprehension. We note that for this study, the notion of visualization considered diverges from Duval's[1]. According to Arcavi (2003, p. 217), *visualization* refers to:

> "*the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings*".

Thus the meaning of visualization is limited to the use of figures, images and diagrams, which are produced in the graphic register. For Duval (1999), however, visualization can be produced in any register of representation, as it is referred to processes linked to the visual perception and then to the vision, which is not limited to only one register. For this study, this notion is not operative enough, being too broad. In spite of it, Duval (1995, 1999) emphasizes some characteristics that distinguish visualization from vision that will be useful for our purpose.

From the characterization of visualization in the context of problem solving, we find it very useful to point out the difference between visual and non-visual methods established by Presmeg in her research about preference to visualize

[1] Duval characterizes the visualization as "a bi- dimensional organization of relations between some kinds of units. Through visualization, any organization can be synoptically grasped as a configuration" (Duval, 1999, p. 15).

(Presmeg 1985, 1991, 1994, 1995, 2006). A *visual method* of resolution involves visual images, with or without a diagram, as an essential part of the solution method. A *non-visual method* of resolution is the one that does not involve visual images as an essential part of the solution method.

Now, the discussion will focus on the differences between the languages used by Presmeg and Duval. For example, the following equivalence should not be made: visual method (Presmeg) - the use of graphic register (Duval); non-visual method (Presmeg) - without using the visual register (Duval). This equivalence cannot be established for two reasons. Firstly, when Presmeg talks about visual images, she includes mental images that belong to the world of mental representations: these are very different from the semiotic representation for Duval (Duval, 1995, p. 14). Secondly, the use of the graphic register does not imply that the method is visual. When Duval (1999, p. 14) reflects about the differences between the way of watching in vision and in visualization, he distinguishes two types of functions for the images: the iconic and the heuristic. *"Iconic representations refer to a previous perception of the represented object […] whereas visualization consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it"* (Duval, 1999, p. 14). Consequently, the visualization is related to a global apprehension of the images and to a conduct of abduction that guides the deduction (Duval, 1995, 1999). Thus, this kind of function performed by these so-called *heuristic* images is found typical of visual methods. As a result, this kind of function typically takes part in the visualization processes as well. Therefore, the use of graphic register does not always imply the presence of visualization. It could lack the global apprehension or the image could perform an iconic function.

From a teaching point of view, the acquisition of this global vision poses a big challenge. Frequently, students do not go further than having a local apprehension and do not see the relevant global organization (Duval, 1999, p. 14). The different levels of apprehension of the geometric and graphic figures (local and global) are needed to develop the conduct of abduction, that is linked to the interaction between a question of mathematical order and the realization of treatment pertinent to this question (Duval, 1995, p. 153). This mental process that is present in mathematical thinking of the experts is not always present in new learners. However, it could be developed with a specific training (Duval, 1999).

Moreover, these are not the only challenges that a specific teaching of visualization could present. Eisenberg and Dreyfus (1991) identified three reasons to explain the reluctance of the students to visualize: "*a cognitive one (visual is more difficult), a sociological one (visual is harder to teach) and one related to beliefs about the nature of mathematics (visual in not mathematical)"* (1991, p. 30).In this paper, and with the help of Duval's approach, it will be possible to describe deeply this higher cognitive difficulty of visualization.

The review of these theoretical elements has enabled us to clarify the meaning and relevance of the research questions. Particularly, the following fundamental notions for the analysis of the results of this study have been defined: visual comprehension, improvement of the learning, visualization, iconic/heuristic function of the images, visual/non-visual methods and some difficulties related to visualization.

3. Methodology and population

The study group was composed of a first year group of 29 Mathematics students at Universidad Complutense de Madrid (UCM), 15 female and 14 male. These students were enrolled in the subject called Real Variable Analysis[2], wherein the formal definition of the integral is introduced. However, this is not the first time they have come across this concept. In high school, they were supposed to have learned the basic rules for the calculation of primitives as well as the relation with the calculation of some areas under curves. This gap in the teaching of the concept between secondary school and university could be the cause of some students' difficulties with the understanding of the integral. However, as it was mentioned in the introduction, the focus of this study is cognitive. Therefore, we do not consider relevant to give more institutional details.

This research has an exploratory, descriptive and interpretative character. These characteristics take part – and are specific - of the qualitative research (Glaser & Strauss 1967; Latorre, Rincón & Arnal 1996). For data collection, a questionnaire of problems and semi-structured interviews have been used. Data analysis is mainly inductive, as categories and interpretation are built from the obtained information. Systemic networks for the questionnaire and transcriptions for the interviews are the main instruments employed in doing this data analysis.

Research instruments

The questionnaire of Mathematical Analysis problems

The questionnaire is composed of 10 non-routine problems in Mathematical Analysis, some of them used in previous research (Mundy, 1987; Llorens & Santoja, 1997; and works quoted in Eisenberg & Dreyfus (1991): Selden, Mason & Selden, 1989 and Tufte, 1988). Most of the problems are posed in an analytic register. However, according to Eisenberg and Dreyfus (1991) *"each of these problems is based on a concept that has a visual interpretation"* (Eisenberg & Dreyfus, 1991, p. 26). Thus the problems allow the study of the students'

[2] In Spanish, *Análisis de Variable Real.*

behaviour with the coordination of registers and they enable us to compare results between those who make a conversion to the graphic register and those who do not. This is possible because each problem admits several kinds of resolution, including a visual one. Therefore, these problems seem to be valid in the study of visual comprehension and students' preference to visualize.

For questionnaire analysis, a recount attending to four indicators was used together with a more qualitative study through *systemic networks* and resume tables, whose results will be showed in the next section (see 4.1.).

The use of systemic networks (Bliss, Monk & Ogborn, 1983) favours a data configuration that allows us to simultaneously look at all effective answers of the students. This way of presenting the data offers a global view interesting to our objectives as it highlights the complex relations between the different elements represented:

- resolution methods of the students and frequency with which they appear (successful or not, visual or non-visual, using graphic or analytic register);
- kind of representations associated to the main concepts used by the students, including the graphical representations.

Nº	PROBLEM'S STATEMENT	DESCRIPTION
1	What's wrong in the following calculation of the integral? $\int_{-1}^{1}\frac{1}{x^2}dx=\int_{-1}^{1}x^{-2}dx=\left.\frac{x^{-1}}{-1}\right\|_{-1}^{1}=\left.\frac{-1}{x}\right\|_{-1}^{1}=\frac{-1}{1}-\frac{-1}{-1}=-2$	Command of the concept of integral and its properties. It is not routine because of the formulation (to look for a mistake instead of calculating)
2	Evaluate $\int_{-3}^{3}\|x+2\|dx.$	Command in the calculation of integrals, with the added difficulty of the absolute value.
3	If f is an odd function in $[-a,a]$ calculate $\int_{-a}^{a}(b+(f(x)))dx$	Command in the calculation of integrals, greater degree of abstraction that requires the use of reasoning and more general properties than in the others.

Table 1: Statements of the analyzed problems and their individual brief description.

Four out of ten problems in the questionnaire involve the concept of integral. In this paper, results of three of these problems completed with data from the interviews are described (see section 4.1.). In the table above (Table 1), the statement of each problem is shown with a brief description about the way in which the concept of integral is involved and some a priori difficulties the students could have. The fourth problem was not included here because it had a low rate of answers (16 blank answers).

The interviews

The results obtained from the questionnaire required a deeper study of the individuals in affective, cognitive and sociocultural aspects related to visualization. In order to do this, semi-structured interviews were conducted with 6 individuals chosen to be representative of different student profiles established according to their preference to visualize and their achievement in the problem questionnaire. The interviews, with an average duration of 90 minutes, were divided into four parts: background of the individual, tasks, beliefs about visualization and queries about the problem questionnaire.

This paper focuses on the students' cognitive difficulties related to visualization. Therefore, we will not deal with all the details about the classification in student profiles and the results of the interviews (see Souto, 2009; Souto & Gómez-Chacón, 2009). This will explicitly deal with the answers of a student to a task based on the Young Inequality (see section 4.2.). In Figure 1, the task is shown. Among other things, the interpretation of a graphic and its posterior connection with the statement of the theorem is asked. This case has been chosen because of its own interest and not as representative of what happened in the other interviews.

1. What do you see in the image?

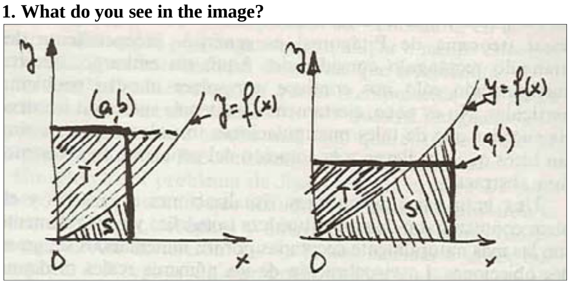


Figure: Image of the Young's Inequality (Guzmán, 1996:22)

2. Do you find some relation with the following statement?

Theorem 3.2 (Young's Inequality). *If f has a continuous and positive derivative on* $[0,c]$ $(c > 0)$ *and* $f(0) = 0$, *then for* $a \in [0,c]$, $b \in [0, f(c)]$, *we have*

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx. \qquad (3.3)$$

The equality holds if and only if $b = f(a)$.

Figure: Statement of Young's Inequality

Figure 1: Statement of the Young's Inequality task of the interview.

4. Analysis and discussion of the results

The following is a description of results of the study that shed some light to the conclusions of the research questions through:

- identification of aspects related to the students' understanding of the concept of integral: difficulties, mental blocks and misinterpretations as well as representations, treatments and conversions that appear in the students' answers. Particular attention is given to students' use of the graphic register;
- analysis of two examples from the data that illustrates two characteristics of visualization: In the first example, there is visualization, as the image performs a heuristic function while in the second one, there is no visualization.

4.1. Difficulties, mental blocks and misinterpretations in the resolution of problems about the concept of integral

Initially, the results of the qualitative analysis of students' answers to the three problems are presented. The overall vision offered by the recount made attending to four indicators follows next.

Problem 1

This problem enables us to explore students' understanding of the concept of the definite integral (that is defined if and only if the function is bounded in the interval of integration) as well as its properties (e.g., positive functions have positive integral). The way and the moment in which the problem is posed are non-routine. Instead of asking for the evaluation of an integral, one is given with mistakes and the student is asked to justify what is wrong with it. Besides, improper integrals have not been explained yet at the moment in which the questionnaire was applied. This problem can be answered either by using the graphic register (by representing the function $f(x) = 1/x^2$ and identifying the area with integral) or the analytic register (by showing that f is not defined and/or not bounded in $x = 0$).

The answers of the students to this particular problem are reflected in the associate systemic network (Figure 2). Valid answers are placed above in the figure and there are only 8 out of the total of 28 non-blank answers. The rest of responses, a total of 20 placed below in the figure, were all in the analytic register. Among them, 7 out of 29 students responded that there was nothing wrong with the calculation. Three of them even repeated the same calculation. The rest (13 students) also tried to repeat the calculation or indicated how it should be done correctly suggesting one or several of the following misinterpretations:

– use of a different primitive;
– interchange between the signs when applying the Barrow rule (In this case, the result obtained is 2, that is coherent with the integration rules.);
– consideration of the constant of integration;
– miscalculations.

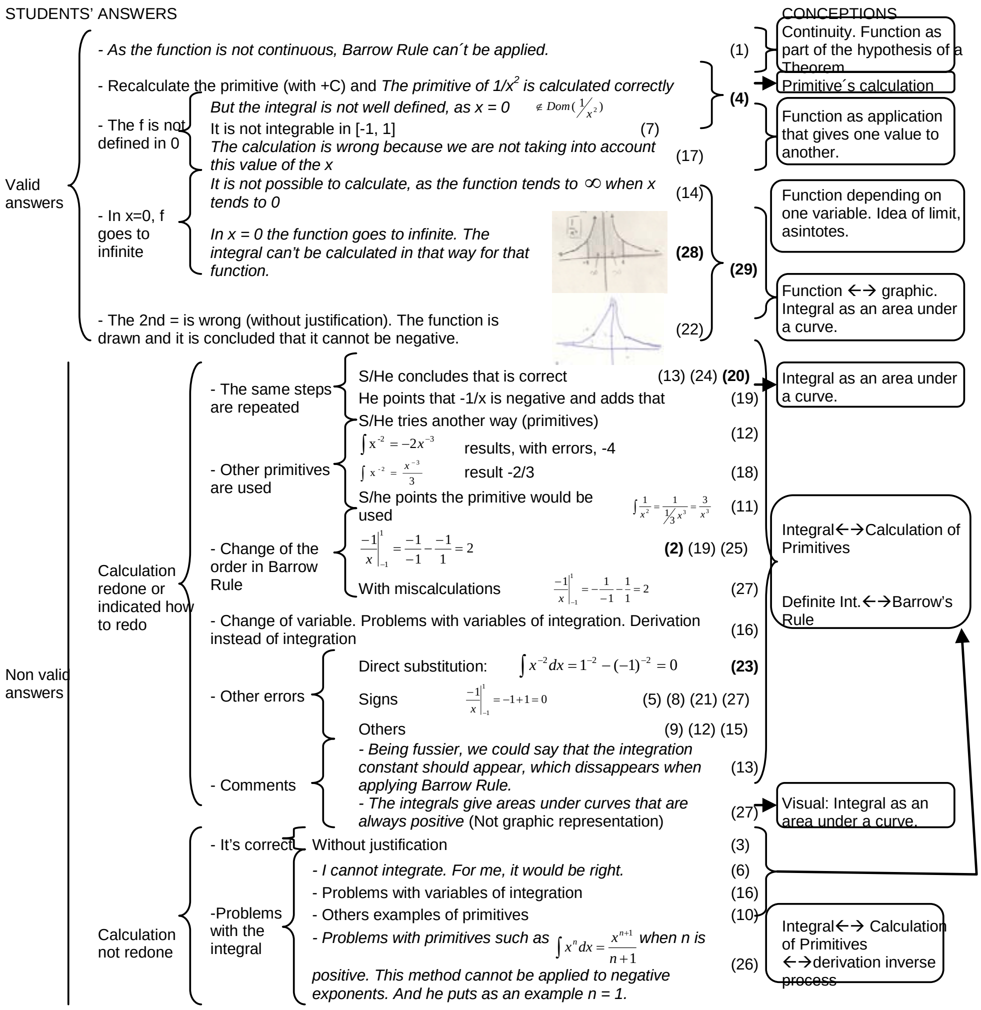


Figure 2: Systemic network associated to Problem 1.

Firstly, it is significant that only 2 out of 29 students explicitly used the graphic register, making a representation of the function and answering correctly. This fact reinforces our hypothesis about the advisability of the use of the graphic register coordinated with the analytic one or at least, the advisability of having a visual idea about what is happening. However, this visual interpretation of the integral can lead to some problems that could be due to the mechanical learning of this connection instead of having the visualization of an image (none of them included a graphic representation in their answer):

- *The integrals give areas under curves that are always positive (27[3]).*
- *As -1/x is negative, when we make Barrow we should change the signs (19).*

Secondly, looking at the balloons on the right side of the systemic network (Figure 2), it is possible to see the importance of the choice of representation in solving the problem successfully. This choice is directly related to the way the integral is being conceptualized. All the students who gave valid answers, placed on the top of Figure 2, either focused on the function (overall properties as continuity or asymptotes, or local at x = 0), or they contemplated the integral as a product (area under a curve). This led two of them to the use of the graphic register. However, most of the students, a total of 22, interpreted the integral as a process (calculation of primitives and Barrow rule or integration as inverse process of derivation) which led all of them to the use of the analytic register. In this case, students do not seem to achieve a complete understanding about what is happening and were not able to give a correct answer, making some of the mistakes cited above.

Thirdly, we examine the way in which representations are used. The data above show that the kind of representation chosen does not completely determine the success or lack of success in the resolution. The way in which these representations are used is also important. For example, student 4 is the only one who, at the beginning, puts attention in the integral as calculation of primitives but answered successfully. This is possible because of the flexible combination of this calculation with another argument about the domain of definition of the function.

Finally, the analysis made from the systemic network associated to this problem (Figure 2) has enabled us to observe how, according to Duval's theory, the appropriation of a diversity of representations of the concepts involved in a problem together with an appropriate and flexible use of them, provide a good understanding of the concepts as well as the problem (see 28 and 29). Although as earlier noted, it is important that this coordination is not made mechanically as it could result to some mistakes (19 and 27). It should be done with reflection.

[3] It was adjudicated a number from 1 to 29 to each student that answered the questionnaire. In the systemic network, this number appears between parentheses together with the answer given by this individual. So, from now on, we will use this numeration to refer both the answer and the students.

Problem 2

In this problem, a direct evaluation of an integral was asked. However, the non-routine aspect of it, as well as its main difficulty, was the kind of function integrated: an absolute value defined in intervals. Therefore, this problem enables us again to explore students' ability in the computation of integrals (application of properties, calculation of primitives, use of Barrow rule, graphic evaluation of integrals). As shown below, this problem admits several kind of resolution using the graphic register aside from the following analytic solution:

$$\int_{-3}^{3}|x+2|dx=-\int_{-3}^{-2}(x+2)dx+\int_{-2}^{3}(x+2)dx=-\left(\frac{x^2}{2}+2x\right)\Bigg|_{-3}^{-2}+\left(\frac{x^2}{2}+2x\right)\Bigg|_{-2}^{3}=...=13$$

Students' answers to this problem are represented in the systemic network of Appendix 1. In this problem, 26 out of 29 students tried, but only 7 were able to solve it. As predicted in the prior analysis of the problem to separate the integral in intervals, this resulted to several difficulties summarized in the following table (Table 2):

KIND OF DIFFICULTY	DESCRIPTION
Problems with the property $\int_a^c f(x)dx=\int_a^b f(x)dx+\int_b^c f(x)dx$	- It is not separated in parts. - It is calculated only one of the parts. - Two parts are considered: o But only the expression of the function is changed, not the integration intervals. o But they are separated at x=0. o But, although they are well calculated separately, they are not added in the end.
Miscalculations	- Primitives - Barrow rule - With operations

Table 2: Students' difficulties with Problem 2 in the analytic register.

The use of the graphic register is more present in students' solutions of this problem than in the previous one, in spite of not being present in the statement of the problem either. Moreover, it appears mainly among those students who answered the problem successfully: 5 out of 8 students who used the graphic register satisfactorily solved the problem. Two of them and 2 made some wrong interpretations although the answers were coherent and comprehension of the problem could be demonstrated.

Therefore, it is possible to analyze with more detail than before students' use of the graphic representations when solving problems. In order to do that, it is going to be useful to take into consideration the difference between iconic and heuristic function of images referred in Theoretical Foundation section. Some differences between both functions of images can be pointed out by looking at some examples. Thus, we can compare the answers of Students 18 and 7 with those of Students 28 and 22, shown in the Table 3.

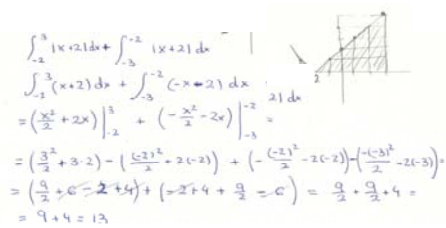
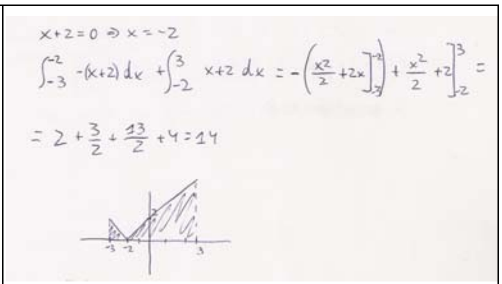
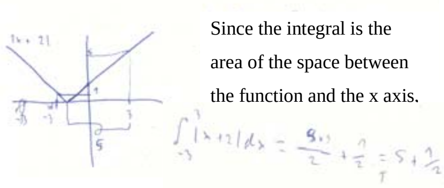
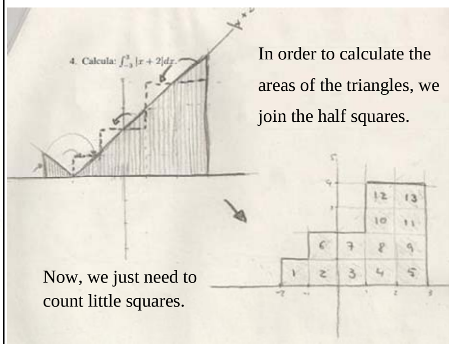
Iconic function	 **Figure 3:** Answer of Student 18.	 **Figure 4:** Answer of Student 7.
Heuristic function	 **Figure 5:** Answer of Student 22.	 **Figure 6:** Answer of Student 28.

Table 3: Iconic and heuristic functions of images in the answers to Problem 2.

All the graphic representations made are almost the same (with or without little squares). However, they have a very different role in each resolution of the problem. In the first row, we can see two answers in which the main argument is analytic accompanied by an image. These images seem to be used either to have an overview of the problem (Figure 3) or to check (Figure 4). However, in the second row (Figure 5, Figure 6), the image is key in the argument. It is essential. That is, if the image was removed, the argument would not have any sense.

Another difference between both functions of the images is that in the iconic answers, there are no graphic *treatments.* This is not necessarily the case in the heuristic ones.

Problem 3

This problem is again about the evaluation of an integral. However, in this case, the degree of abstraction is higher as the function to be integrated is not completely known (it is only known that it is odd) and it depends on a parameter, *b*. Thus, difficulty increases and a deeper understanding of the fundamental concepts involved is needed. These concepts are the odd function and the definite integral. Both of them admit either a graphic or an analytic representation, being any of them suitable in solving the problem.

We note a particularity of the statement of this problem in relation to previous ones. The fact of being *f* defined as "odd", instead of writing for example, $\forall x \in [-a,a] \quad f(-x) = -f(x)$, it could suggest the use of the graphic register or at least, the coordination of registers. However, the conversion to the graphic register is not compulsory to solve the problem. Actually, although the use of the graphic register in this problem is higher than in the previous ones, the strategy followed by most of the students (12 out of 20 who tried to solve the problem) is eminently analytic and it is the following:

$$\int_{-a}^{a} \left(b + \left(f(x)\right)\right)dx = \int_{-a}^{a} bdx + \int_{-a}^{a} f(x)dx = bx\Big|_{-a}^{a} + 0 = 2ab$$

Students' answers to this problem are presented in the systemic network of Appendix 2. There, it is possible to see that only half of the students that used this analytic answer give the correct answer.

KIND OF DIFFICULTY	DESCRIPTION
Problems with the definition of odd function	- Not to know how to use it. - To change it with the definition of even function. - Punctual definition, only in the extremes. - To use explicitly or implicitly: f(x)+b odd
Problems with the defined integral	- Calculation: similar to previous problems. - Properties: o Additive: mental block if is not applied at the beginning. o To take out of the integral additive constants. o Not to change the sign when a change either in the integration extremes or in the variables is made.
Symbolic misinterpretations	- Problems with notation (names of parameters) Problems with the integral symbol, (used to refer the primitive) $\int_{a}^{b} \left(f(x)\right)dx = \int f(a) - \int f(b)$

Table 4: Difficulties of the students with Problem 3 in the analytic register.

The rest of strategies followed consist either of separating the integration interval at 0 or having a more abstract character, producing some logical misinterpretations. For example, Student 13 tried to reason out by proving that *f* even implies *f'* odd and then concludes that *F*, the primitive, is even. That is, he proved an implication that is used in the opposite sense afterwards. In Table 4, the main difficulties found in analytic answers are summarized. Among these difficulties, we find some of the common difficulties of the calculation of integrals pointed in previous problems aside from those predicted on the a priori analysis due to the abstract character of the problem. New obstacles related to the use of more general arguments have been added too. Moreover, in this case, the concept of integral appears intertwined with the concept of odd function, which presents its own difficulties.

In relation to the graphic register, there are 8 out of 20 students who used it in their answers. Taking into consideration again the difference between iconic and heuristic function of images, the use of graphic register in this problem presents some particularities that are worth underscoring.

Firstly, we found the most common type of answers, 5 out of 8 students, in which the graphic representations used appear in combination with the analytic register. In these cases, images are employed either to try to remember the definition of odd function or to obtain some other properties, but they do not contribute very successfully in the resolution of the problem. Therefore, image is not essential and it could be said it performs an iconic function. Thus, we say that the method of resolution is *non-visual¸* in spite of the graphic register's presence.

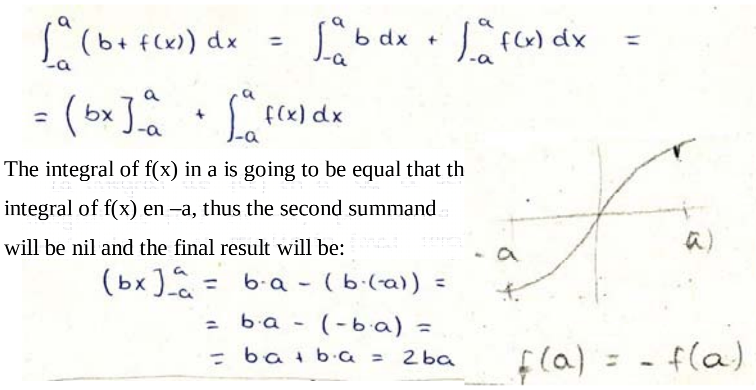


Figure 7: Non-visual answer of student 20 to Problem 3.

The solution above (Figure 7) illustrates this kind of use of the graphic register. It belongs to Student 20, the only one among five students who is able to give the correct answer. The main argument is analytic and it appears with a graphic representation, which is interpreted by giving an incorrect definition of odd function. However, this wrong interpretation did not affect the analytic reasoning, which is valid. Moreover, this example shows one difficulty typical of iconic use of images: the lack of coordination between graphic and analytic registers. In this case, lack of coordination can be attributed to several reasons: the "pointwise" understanding of the graphic representation (Monk, 1988 quoted in Eisenberg & Dreyfus, 1991), or the lack of connection between integral and area.

Secondly, there are answers in which images perform a heuristic function (3 out of 8 students). Two very different arguments are found among them attending to a different number of conversions made between the analytic and the graphic registers. The first kind of argument was given by only one student (Figure 11). It is completely *visual* since the main argument is developed in the graphic register. Therefore, only one conversion is made (analytic-graphic). This answer will be analyzed more extensively in the next section (see 4.2).

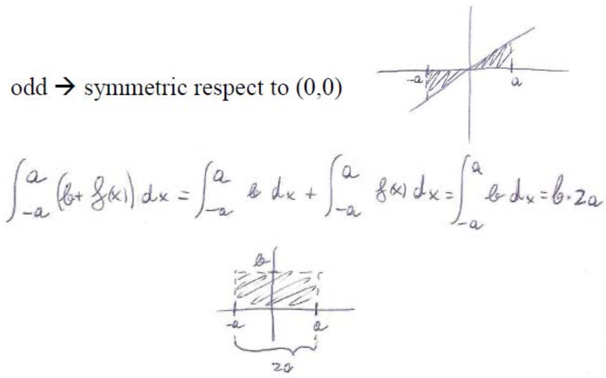


Figure 8: Mixed answer of Student 7 to Problem 3.

The second kind of argument was used by two students, one of them (student 7) giving a valid answer (Figure 8). As a first step in the analytic register is needed in order to apply the additive property of integrals, it has been called *mixed*. Later on, each integral is converted into a graphic representation which allows calculating their value without performing treatments in the algebraic register. Finally, it is needed to revert to the analytic register in order to finish the evaluation of the integral. Thus, in this last case, two conversions are made (analytic-graphic-analytic). In both arguments, images are essential but in a very different way.

Overall view of the results of the questionnaire.

In order to have an overview of the results of the questionnaire, a recount of students' answers attending to the following four indicators was made:

- Ans- Number of students that do not leave the answer blank.
- Corr- Number of students that answer correctly.

- VA- Number of students that use a visual method.
- GR- Number of students that make some graphic representation.

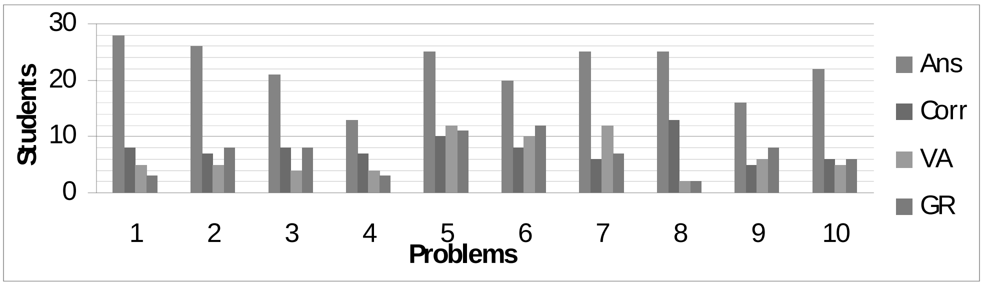


Figure 9: Graph of the overall results of problems.

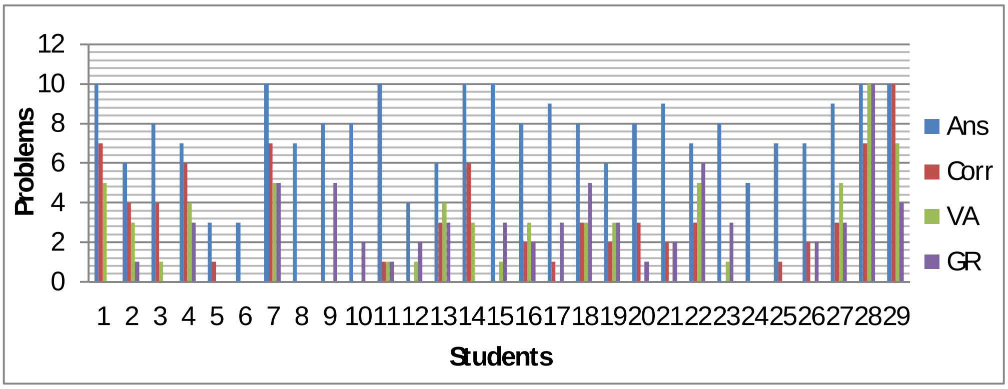


Figure 10: Graph of overall results of students.

In Figure 9 and Figure 10, each column corresponds to one indicator. The relation between the first two columns (Ans and Corr) enables us to have an idea of students' achievement in the questionnaire. The relation between the last two columns gives us an idea of the presence of visualization processes in students' answers, making the distinction between the presence of a graphic representation (GR) and a visual argument (AV). As it has been discussed above, this distinction is related to the difference between iconic and heuristic function of images. Therefore, the limited success and presence of visualization processes in students' answers are pointed out. This conclusion is valid for both, the problems related to the integral concept (1, 2, 3 presented in this paper and 4) and the others in the questionnaire[4].

[4] The original ordering in the questionnaire was different and problems about integral were not consecutive. It has been modified here in order to make it coherent with the numbers in the analysis presented in this paper.

Furthermore, results can be interpreted from two points of view offering different perspectives: (1) interpretation from the problem point of view gives us an idea about the students' understanding of the main concepts of Mathematical Analysis, particularly that of the integral; (2) interpretation from the student point of view gives us an idea of individual differences according to achievement and preference to visualize.

4.2. An example and a counterexample of visualization

Until now, data has shown how the prevalence of the analytic register as well as the lack of coordination between registers. Both results explain some of the students' difficulties in solving problems on the concept of integral. Following our initial hypothesis, special attention has been given to the graphic register. It has been found that its use cannot be always considered as visualization, as it is related to the heuristic function of images. But how can we help students to use images in a heuristic way? Now, we will analyze two examples taken from the data in order to illustrate two characteristics of visualization that should be taken into account in its specific teaching: a high cognitive requirement and the need of a global apprehension of the images.

Analysis of a visual method of resolution: a high cognitive requirement

The first example we are going to analyze is the visual argument found among the answers to Problem 3 (Figure 11). It belongs to Student 28, which shows (Figure 10) a clear preference for visualization and an acceptable achievement in the questionnaire (7 correct out of 10 answered problems). The analysis of this original visual answer enables us to go deeper into the cognitive difficulties of visualization pointed out by Eisenberg and Dreyfus (1991).

As it was noted above, the main argument is visual because it takes place completely in the graphic register. In the end, there is also a non-visual answer that takes place in the analytic register. It could be considered as verification more than as the main argument. This non-visual answer follows a linear process consistent in the succession of several algebraic treatments performed without mistakes giving *2ab* as the result. Unlike the non-visual answer, the visual one is far from being linear. Actually, more concepts and relations should be taken into account simultaneously. These are the following: (1) It is required a visual interpretation of the integral as an area and of the odd functions as those symmetric with respect to the origin; (2) It should be remembered that adding a constant quantity to a function means a translation of its graph along the y-axis though this notion is not necessary for the analytic answer; and finally, (3) There is the need to generate areas through some editing similar to "cut and paste", considering their signs.

Obviously, in both arguments, the result is the same. However, each kind of argument leads us to see the problem in a very different way.

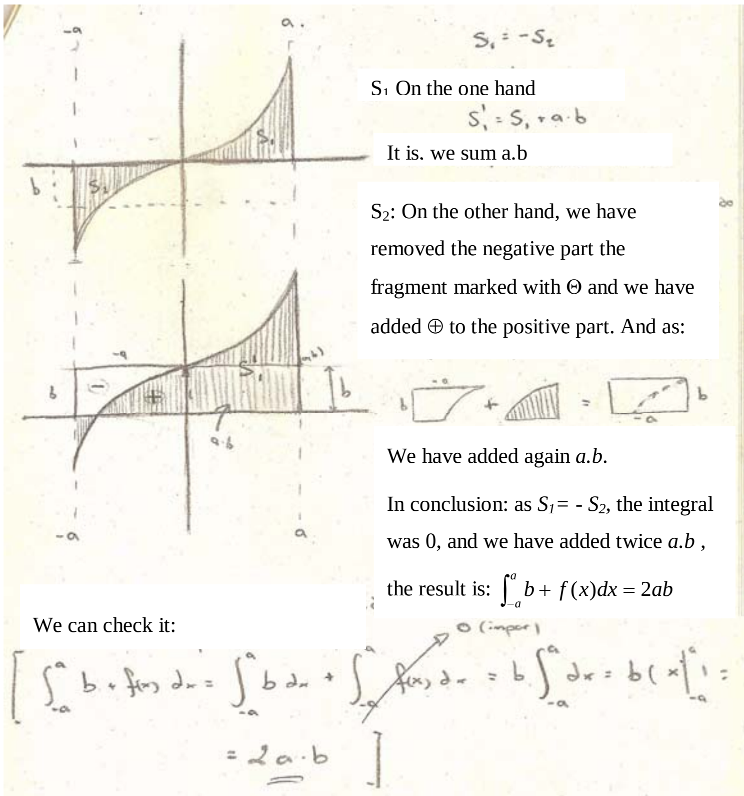


Figure 11: Visual answer of Student 28 to Problem 3.

Therefore, each register provides a very different understanding of the situation but they require different cognitive processes. In particular, in this example, it has been found that visual argument is not linear and it involves more relations and notions than in the analytic one (for example, the translation along the y-axis). Thus, from a didactic point of view, we emphasize the convenience of combining both kinds of arguments, visual and non-visual, but taking into consideration the higher cognitive requirement of the former.

The case of Silvia: The global apprehension

The following discussion is obtained from the interviews. Unlike the previous example, visualization as defined in this study seems to be absent. Nonetheless, the analysis of this case is illustrative. As counterexamples in Mathematics, this analysis makes us know what is not working in this case and realize that what is occurring here is useful in understanding the aim of the study, that is, processes of visualization.

The subject chosen for this interview, named Silvia (Student 23), was selected because her responses to the questionnaire showed some preference for graphic register (see Figure 10). However, her achievement in the questionnaire was low: (no correct answer in 8 problems tried). Therefore, some difficulties related to visualization were conjectured. The interview enabled us to go deeper into these difficulties. There are three notable parts in these interviews corresponding to the task based on Young's Inequality.

The first part belongs to the moment in which only the image (Figure 1) is shown to her and she was asked to interpret it. Silvia detected isolated elements (1[5]) and even made some references to the integral as an area (7). She based the difference between the two images in Riemann's theory of integration, which has been taught recently. But there is no clear evidence if the areas she has identified are the ones involved in the inequality.

The second part begins when the statement of the inequality is shown (Figure 1) and she was asked if there are some relations with the image. She presumed a relation, but it does not seem to be too clear for her (8). Actually, she frequently asked for help in making questions (12, 17, 18). Finally, it seemed that she has found the graphic interpretation of the statement when she stated: *"The addition of the areas contained by the function is bigger than the area of the rectangle"* (14).

However, later in the interview, she admitted that she did not understand well the graphic representation ("drawing"): *"Well, to understand it [the theorem], with the drawing I wouldn't understand it*" (besides 21, 22). In this last part, in order to be able to continue with the interview, some indications were given to the student in order to help her to correctly identify all the elements in the image with those in the statement (from 23 to 32). Silvia seemed to be able to coordinate both registers. Nevertheless, there was not any clear moment of understanding (33, 34).

Several reasons for explaining this lack of understanding could be pointed out: her incapacity of seeing f^{-1} in the image (17, 18), the way integral has been recently explained in class (7). However, from the point of view of visualization, the main hypothesis established is the lack of global apprehension. Thus, for her, the image was only an illustration, an iconic representation that does not work as a means of visualizing the statement of Young's Inequality.

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[5] This number makes reference to the place occupied for the fragment of transcription of the interview referred in the text. This number appears in the left column of the transcription in Appendix 3.

These two pieces of data allow us to show the high cognitive difficulty of visualization and the necessary but not sufficient condition of coordinating two different representations of the concept of integral in order to visualize. In this way, it is possible for the student to establish a relation between these two representations but it could be not enough to make an image meaningful, that is, to overcome the iconic function of the graphic representation. Global apprehension is needed as well.

5. Conclusion

In this paper, the following two research questions formulated in the introduction have been explored through a theoretical discussion and the analysis of empirical data obtained in a larger study:

- What kind of understanding of the concept of the integral do university level students possess and exhibit?
- What are the possibilities that visualization offers to improve the students' learning of the concept of the integral?

An initial hypothesis based on previous research (Mundy, 1987; Llorens y Santoja, 1997; González-Martín & Camacho, 2005) was behind these two questions: first year undergraduate students' mechanical and operative way of using the concept of integral could be attributed to the lack of integration between the concept of area and the integral and therefore, the lack of coordination between the analytic and graphic registers. This hypothesis lead us to take into account the cognitive theories of registers of semiotic representation (Duval, 1995, 1999, 2006; Hitt, 2003, 2006) in order to explain and analyze students' difficulties with the concept of integral. Research on visualization (De Guzmán, 1996; Arcavi, 2003; Presmeg, 1985, 1991, 1994, 1995, 2006; Eisenberg & Dreyfus, 1991) has been useful to analyze students' productions too, incorporating other perspectives of analysis that go beyond the semiotic approach.

With this point of view of our problem, in order to answer the first research question, we have identified aspects related to the students' understanding of the concept of integral paying special attention to students' use of the graphic register: difficulties, mental blocks and misinterpretations as well as representations, treatments and conversions that appear in the students' answers to three problems of the questionnaire. In order to explore the second research question, two examples from the data have been analyzed. This is to illustrate the characteristics of visualization that should be taken into account in its specific teaching.

In this analysis, it has been found of particular interest to exploit the distinction between *iconic* and *heuristic function of images* (Duval, 1999). Moreover, the *heuristic function* has been identified with *visual methods* (Presmeg, 1985) which inspired us to distinguish three different kinds of methods in solving a problem using the graphic register: *non-visual, mixed* and *visual*. Therefore, visualization, as being dealt with in this study, is not necessarily always present when graphic register is used.

Other important conclusions in this paper are: (1) Operative and mechanical students' understanding of the concept of integral, except in some individual exceptions; (2) Teaching visualization by combining visual and non-visual aspects is a complex task in which specific characteristic of this ability should be taken into account.

1. *Operative and mechanical students' understanding of the concept of integral, except in some individual exceptions.*

According to our initial hypothesis, the analysis of the three problems highlighted the prevalence of an operative and mechanic use of the integral which appears to be linked to the use of analytic register and which results to numerous miscalculations. This result is characteristic of the concept of integral but also of other concepts of Mathematical Analysis, as the overall view of results pointed out. Some reasons that may explain this fact are the following:

a. *The connection between the choice of representation and the conception of integral owned:* It was noted in the analysis of the systemic network associated to the Problem 1 that integral seen as a process leads to the identification with the notions of "primitive" and Barrow's rule and hence, the choice of an analytic representation. Nevertheless, integral seen as a product leads to the identification with the notion of area and consequently, to the choice of a graphic representation as well. Most of the students have a process point of view of the integral.

b. *The lack of flexible coordination between the analytic and the graphic register*. As it has been pointed out, most of the students choose the analytic register and eventually, they got stuck into it. There are no evidences of that they are able to integrate the concept of the integral and area. However, this connection has been found advisable either to check the answer (see answer of the student 28 to Problems 3) or to deal with the non-routine character of the problems (see analysis of the Problem1, students 28 and 29).

c. *Iconic versus heuristic function of images*: On the other hand, among those students who use the graphic register, we frequently find that the image remains as an accessory element in the resolution of the problem, not being as meaningful as it might be. That is, there is a frequent iconic use of images, which does not contribute to get a better understanding of the situation. This happened, for example, in the answer to Problem 3 of the Student 20 (Figure 7). Another consequence of this iconic use of images is a small number of substantial examples of image treatment, such as the answer of the Student 28 to Problems 2 and 3 (Figure 6, Figure 11). Image treatment has been found related to the heuristic function of images.

d. *The problem of conversion and the importance of the register of problem's statement:* Most of the problems were posed in an analytic register. Therefore, a conversion was needed in order to solve them by using the graphic register. Thus, the shortage of visual answers can be explained by saying that conversion is difficult for students, which confirms results of previous research (Duval, 1999). This difficulty lies between cognitive and sociological reasons for reluctance to visualize pointed by Eisenberg & Dreyfus (1991). If only analytic register is used in classroom, as conversion is difficult for students, they are going to be condemned to use it.

2. *Teaching visualization by combining visual and non-visual methods is a complex task in which specific characteristic of this ability should be taken into account.*

The convenience of combining both kinds of arguments, visual and non-visual, is pointed out. The analysis of the first example of the section 4.2 highlighted that each register provides a very different understanding of the situation. But how can we help students to use images in a heuristic way? Some characteristics of visualization should be taken into account:

a. *The flexible coordination of both registers:* It involves having at least a representation in each one, to identify the represented units and to find relations between them in both representations. We claim that the coordination of registers can be learned. Thus, taking into account the findings of this study, we emphasize the necessity of a specific training of image treatment and conversion via tasks proposed in both registers as part of a teaching of visualization.

b. *The global apprehension:* To coordinate an analytic and a graphic representation could be not enough to make an image meaningful. That is, to overcome the iconic function of the graphic representation. As we could see through the analysis of Silvia's episode, global apprehension is needed too. It allow going beyond mere identification of the represented units.

c. *The high cognitive difficulty of visual arguments:* Besides the coordination between the analytic and graphic registers, and the global apprehension, we see in the example of the visual answer to Problem 3 that visual methods are not necessarily linear and that they can involve more relations and notions that the analytics ones (in the example, the translation along y's axis). This higher cognitive difficulty of visual reasoning should be taken into account in a specific teaching of visualization too.

In the future, we find it relevant to continue researching around these three factors as well as how to put them into practice. It could be done through an explicit teaching experiment of the visualization of concepts. In this way and by thinking on the impact of this research into the practice, we believe that the analysis made from the three problems via the systemic networks is a valuable material that can be used in the design of teaching situations for the classroom. For the future, we also find it important to undertake more studies like this in order to make advances in a systematic knowledge of the different kind of representations, treatments and conversions in each register of the main concepts of mathematics.

Acknowledgments

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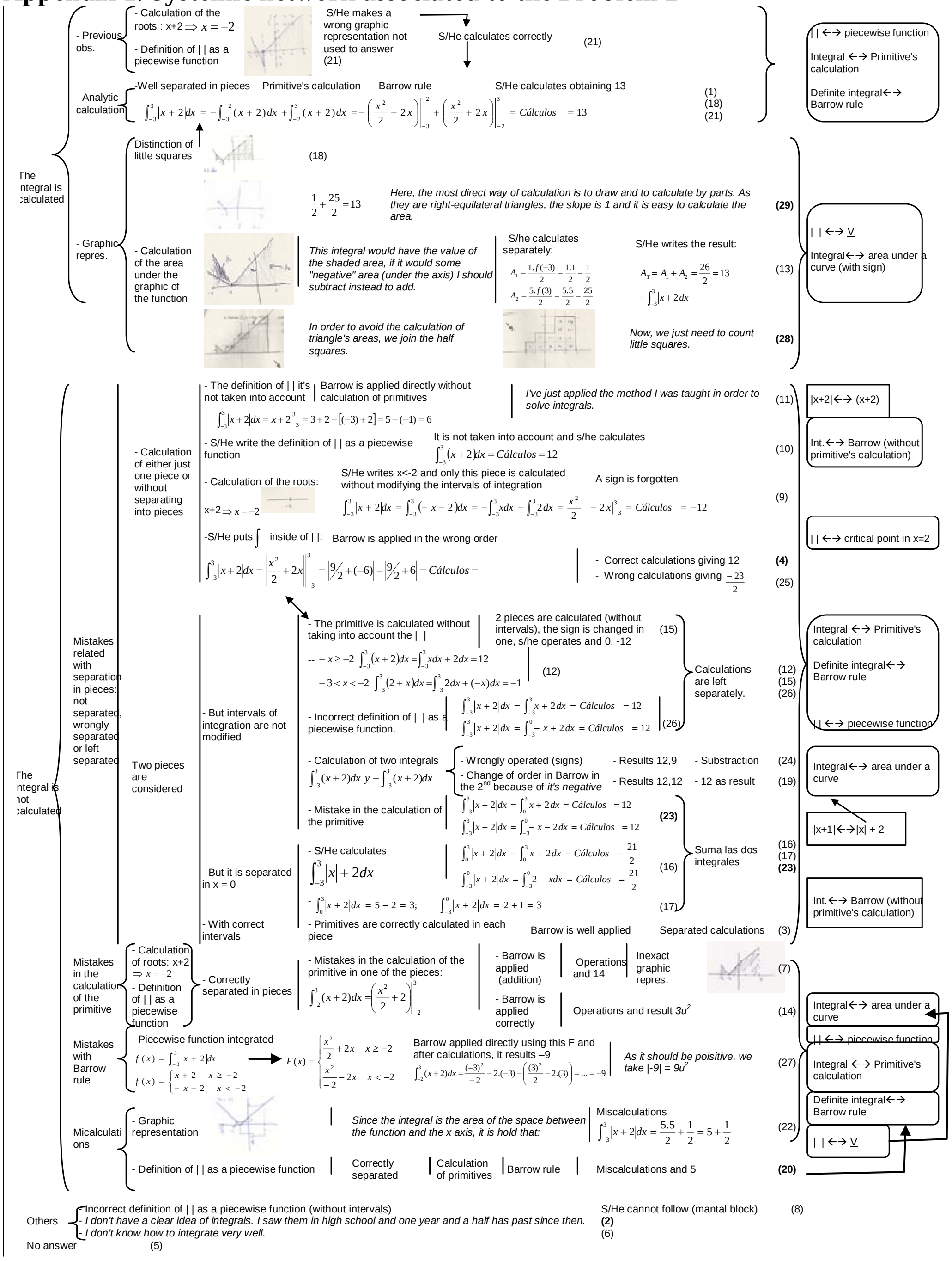
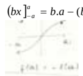
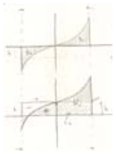
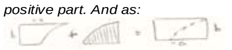
Appendix 1. Systemic network associated to the Problem 2



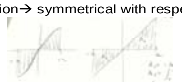
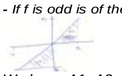
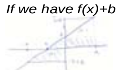
Appendix 2. Systemic network associated to Problem 2

Correct answer

- **Additive property** $\int_{-a}^{a}(b+f(x))dx = \int_{-a}^{a} bdx + \int_{-a}^{a} f(x)dx$
  - **Analytic calculation** $\int_{-a}^{a} f(x)dx = 0$
    - - *Due to the fact that f(x)=-f(-x) for each x [-a,a]* $0=\int_{-a}^{0} f(x)dx + \int_{0}^{a} f(-x)dx = -\int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx = 0$ | $= bx\big|_{-a}^{a} + 0 = ab + ab = 2ab$ (1)
    - -No justification (16)
    - -Because f(-a)=-f(a) (3)
    - -*Because of f being odd* | $= [bx]_{-a}^{a} = 2ab$ (14)
    - $= [bx]_{-a}^{a} + \int_{-a}^{a} f(x)dx$ *The integral of f(x) in a is going to be equal that the integral of f(x) in –a, thus the second addend will be nil and the final result will be:* | $(bx)_{-a}^{a} = b.a - (b.(-a)) = \ldots = 2ba$  **(20)**
    - *Here I had the image of the function y=x, which is a standard of odd function. However, it has not being to see if* $\int_{-a}^{a} f(x)dx = 0$ *but once I knew that, to check it.* | $= 2ab$ **(29)**
  - **Graphic calculation**
    - -*odd⇒symmetrical with respect to (0,0)* |  +  | $= b.2a$ (7)
- **Non additive property**
  - 
  - *S1=-S2*
  - *On the one hand S1'=S1+a.b*
  - *On the other hand, we have removed the negative part the fragment marked with - and we have added ⊕ to the positive part. And as:* 
  - *We have added again a.b.*
  - *In conclusion: as S1= - S2, the integral was 0, and we have added twice a.b , the result is:* $\int_{-a}^{a}(b+f(x))dx = 2ab$
  - *We can check it* (and he makes the analytical calculation like(1)). **(28)**

- Odd ←→ $f(-x) = -f(x)$ $\forall x \in [-a,a]$
- Integral←→ Primitive´s calculation
- Definite int.←→ Barrow Rule
- Odd←→ Symmetrical with respect to (0,0)
- Integral←→ area under a curve (with sign)
- f(x)+b←→ translation along the vertical axis

Non complete or incorrect answers

- **Additive property**
  - -*f odd [-a,a]* | $\int_{-a}^{a}(b+f(x))dx = \int_{-a}^{a} bdx + \int_{-a}^{a} f(x)dx = b\int_{-a}^{a} dx + \int_{-a}^{a} f(x)dx$ (8) — Definite int.←→ Barrow Rule without primitive's calculation
  - -*If f is odd f(-a)=-f(a)* | $\int (b+f(x)) = b + f(a) - (b + f(-a)) = \ldots = 2f(a)$ (11)
  - -f is odd f(-x)=-f(x) | $\int_{-a}^{a}(b+(f(x)))dx = \int_{-a}^{a} bdx + (f(x))dx = \int_{-a}^{a}(f(x))dx$ | f(a)dx (12) — Odd ←→ $f(-x) = -f(x)$ $\forall x \in [-a,a]$
  - $-\int_{-a}^{a}(b+(f(x)))dx = \left[\frac{b^2}{2} + \int f(x)dx\right]_{-a}^{a} = \frac{b^2}{2} + \int f(a)dx - \left[\frac{b^2}{2} + \int f(-a)dx\right] = \int f(a)dx - \int f(a)dx$ | *We know for being odd that f(a)=f(-a) therefore* $\int_{-a}^{a}(b+(f(x)))dx = 0$ (21) — Definite int.←→ Barrow Rule with partial primitive's calculation
- **Odd definition**
  - -*As it is odd* — **In order to be integrable it should be continuous in x=0 and knowing that it is odd; F(0)=0.* $\int_{-a}^{a}(b+(f(x)))dx = 2\int_{0}^{a}(b+(f(x)))dx = 2\left(bx\big|_{0}^{a} + F(x)\big|_{0}^{a}\right) = 2(b.a - b.0 + F(a) - F(0)) = 2b.a + 2(F(a) - F(0)) = 2b.a + 2F(a)$ (27) — Definite int.←→ Barrow Rule without primitive's calculation
  - - $\int_{-a}^{a}(b+f(-x))dx = \int_{-a}^{a}(b - f(x))dx$, $b + \int_{-a}^{a} f(-x)dx = b - \int_{-a}^{a} f(x)dx$ | b's are cancelled and everything comes to the same side | $\int_{-a}^{a} f(-x)dx + \int_{-a}^{a} f(x)dx = 0$, $\int f(-a)dx - \int f(a)dx + \int f(a)dx - \int f(-a)dx = 0$ **(23)** — Odd ←→ $f(-x) = -f(x)$ $\forall x \in [-a,a]$
  - - f(-x)=f(x) in [-a,a] |  | As the function is odd we see that $\int f(-x) = \int -f(x)$ | $2\int_{-a}^{a}(b + f((x)))dx$ (10) — Integral←→ area under a curve
  -  | *f(x)= sen x odd function* *f(-x)= - f(x)* (26) — Odd←→ Symmetrical with respect to (0,0)
- **Graphic calculation**
  - - Odd function→ symmetrical with respect to (0,0)  | $A_1 = -A_2$, $A_1 = \int_{0}^{a}(b+(f(x)))dx$, $A_2 = \int_{-a}^{0}(b+(f(x)))dx = -\int_{0}^{a}(b+(f(x)))dx$ | For being an odd function […] it is hold that $\int_{-a}^{a}(b+(f(x)))dx = 2\int_{0}^{a}(b+(f(x)))dx$ (18) — Integral←→ area under a curve (with sign); f(x)+b←→ even; Odd←→ Symmetrical with respect to (0,0)
  - - *If f is odd is of the sort of*  *We know A1=A2* | *If we have f(x)+b*  | *f(x)+B=0 si x=α* $\int_{-a}^{a}(b+(f(x)))dx = b.a + 2.\int_{0}^{-\alpha} f(x) - b$ (22) — f(x)+b←→ translation along the vertical axis; Integral←→ area ¿without sign?
- **Others**
  - - S/He separetes into parts: $\int_{0}^{a}(b+(f(x))) = b + f(a) - b = f(a)$, $\int_{-a}^{0}(b+(f(x))) = b - b - f(-a) = -f(-a)$ |  | $\int_{-a}^{a}(b+(f(x))) = f(a) - f(-a)$, $f(-a) = -f(a)$ *for being odd*, $\int_{-a}^{a}(b+(f(x))) = 2f(a)$ (17) — Odd←→ Symmetrical with respect to (0,0)
  - $-\int_{-a}^{a}(b+(f(x)))dx = (bx + F(x))_{-a}^{a} = ba + ba + F(a) - F(-a)$ *Being F the primitive of f(x)* (Barrow rule applied) | It is proved that f even⇒ f' odd (chain's rule) *Then F(x) is even* ⇒F(a)=2F(a) | Therefore $\int_{-a}^{a}(b+(f(x)))dx = 2(ba + F(a))$ (13) — Definite int.←→ Barrow Rule without primitive's calculation; Logical reasoning
  - - *I don't know* (6)

Non answer — **(2)(4)**(5)(9)(15)(19)(24)(25)

Appendix 3. Transcription of Silvia's interview

Part 1

1	S: I don't know. Let me see. There's a function (pointing at the function) and they give you a point (pointing at the point) and you calculate the area under (encircling the rectangle), isn't it? Between the function and the point.
2	E: OK. Is there something more that occurs to you?
3	S: I don't know.
4	E: How would you calculate the area?
5	S: The area of… the rectangle?
6	E: Um- huh.
7	S: Well, integrating the function… But still… This is what we're studying. So, integrating the function and… Let me see. This is the superior area of the function (she points with her finger at the upper part of the first drawing) and this is the inferior (she points with her finger at the lower part of the second drawing). I don't know.

Part 2

8	S: So the positive integral will be this. From 0 to a… (She puts her finger on the x-axis). (She quietly thinks for a moment). I don't know. Let me say that this is the same, isn't it? They are representing it to you. Let me see what it's doing here… Which one is the point a, this one?
9	E: Do you see any point in which *a* is written?
10	S: Yes, here, around *a* and *b* (she points at the point *(a, b)*).
11	E: OK.
12	S: Then it makes the area greater than or equal to *ab*. And what's *ab*? The rectangle?
13	E: What do you think?
14	S: Yes, yes. The sum of areas contained by the function is greater than the area of the rectangle.
15	E: And where are the areas contained by the function here in the statement?
16	S: In the integral.
17	E: OK, and this, the f^{-1}?
18	S: This, no? (She points at the second image). I don't know.
19	E: Well, but you find some relation, don't you?
20	S: Yes.

Part 3

21	E: OK, what kind of explanations would you need with the drawing? Have you understood it completely, the drawing?
22	S: Um… Well…
23	E: The continuous function, with a positive derivative. Can you see it?
24	S: Yes, this (pointing at the 1st image and draws it in the air)
25	E: OK. After it passes by the 0, this has been shown, and where do you place the *a* and the *b*?
26	S: Well, the a (pointing at the *x*- axes) and the b (pointing at the *y*- axes).
27	E. OK, and then, now they say to you that ab, what was this? What did you say it was?
28	S: The area… of the rectangle.
29	E: OK, this is less than or equal to this integral, But where is the integral in the drawing?
30	S: Well, it'll be this, from *0* to a (pointing with her finger at an interval over the *x*-axes). This one, the *S*'s.
31	E: OK, and the other?
32	S: Well, *T*'s. (She's thinking).
33	E: This is little more difficult for you to see, isn't it?
34	S: Yes.