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ABSTRACT ALGEBRA LEARNING: MENTAL STRUCTURES, DEFINITIONS, EXAMPLES, PROOFS AND STRUCTURE SENSE

Abstract. A reflection is made about Abstract Algebra Learning, motivated by the reading of an article published previously in this journal. The reflection involves elements of APOS (Action – Process – Object – Schema) Theory and related published works as well as results from other studies involving definitions, examples, proofs and structure sense.

Keywords: Abstract Algebra, group theory, mental structure, definition, example, proof, structure sense

Résumé. Apprentissage de l'Algèbre Abstraite : Structures mentales, définitions, exemples, démonstrations et sens de la structure. Une réflexion est présentée autour de l'apprentissage de l'Algèbre Abstraite, motivée par la lecture d'un article publié dans cette revue. La réflexion inclut des éléments de la théorie APOS (Action – Processus – Objet – Schème) et des travaux publiés en relation avec cette théorie, ainsi que les résultats d'autres recherches incluant définitions, exemples, démonstrations et sens de la structure.

Mots-clés. Algèbre Abstraite, théorie de groupes, structure mentale, définition, exemple, preuve, sens de structure

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1. Introduction

The writing of this paper came about after reading the article of Durand-Guerrier, Hausberger & Spitalas (2015) about the prerequisite knowledge for the learning of Modern Algebra (or Abstract Algebra). Rather than being a *reply* to that article, it can be thought of as a *complement*, and perhaps the latter view would allow us to understand better some results that the authors present in their study.

The intention of this paper is mainly two-fold: On the one hand to offer a non-exhaustive review of studies in order to outline what research has found out about Abstract Algebra learning, and on the other, to discuss some aspects that interact with this learning and that are mentioned in the Durand-Guerrier et al. article, such as the generation and use of examples; the role that definitions play; production of proofs; and structural thinking. In relation with this focus, some results from research in Linear Algebra learning are also mentioned.

Other aspects raised by Durand-Guerrier, Hausberger & Spitalas (2015) are not less important, such as the use of quantification, the relationship between Linear Algebra and Abstract Algebra, and the role of logic in understanding mathematical structures, also motivate reflections, but are left to a possible future article.

ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES, volume 21, p. 297 - 316.

2. Research on Abstract Algebra Learning and Teaching

In 2001 Findell wrote that a literature search using specific criteria revealed 15 articles on the learning of Abstract Algebra; 11 of these were published since 1994, of which 9 originated from the work of Dubinsky, Leron and their collaborators. Since 2001 there have been an increase in this kind of studies, although it can still be claimed that the number of published studies in this area is slim compared to others at the undergraduate level, such as calculus and analysis, as noted by Durand-Guerrier et al. (2015) as well. On the other hand APOS-related work constitutes a major part of the growing body of research (Arnon et al., 2014) on Abstract and Linear Algebra learning and teaching. These studies offer insight into different aspects of the learning process; given the situation, studies carried out using this theoretical framework deserve some credit in an article about the learning of Abstract (Modern) and Linear Algebra.

Concepts such as binary operation, group, subgroup, center of a group, isomorphism, coset, normality, quotient group, permutations and symmetries have been researched and discussed in several publications (Asiala et al., 1997; Asiala et al., 1998; Brown et al., 1997; Dubinsky et al., 1994; Leron and Dubinsky, 1995). In these studies, usually through the design of a preliminary genetic decomposition that indicates a possible way to learn a certain mathematical concept, researchers explicitly state the proposed mental stages involved in constructing it. This is the first step of the research cycle, known as the *theoretical analysis*. After this is done, instructional strategies developed specifically with the purpose of motivating the proposed mental constructions in the genetic decomposition are implemented, constituting the second step of the research cycle. One common strategy in this phase has been the use of computer programming to help with the interiorization of Actions into Processes and encapsulation of Processes into Objects. Finally, data is collected mainly through interviews in order to observe whether the mental path sketched by the genetic decomposition is in line with how students seem to be learning the concept in question. The cycle is iterated until a satisfactory explanation is reached in terms of student understanding; this happens when the theoretical analysis and empirical findings converge.

The mental structure known as Action is constructed when an individual responds to an external cue such as a formula or an algorithm and is applied to previously constructed objects. When they are repeated and reflected upon, they are converted into Processes via the mechanism of *interiorization*. When there is need to apply Actions on Processes, they are *encapsulated* into Objects to which actions can be applied (for more information about APOS Theory and its components, the reader can consult Trigueros & Oktaç, 2005; and Arnon et al., 2014). In this theory, if supported by empirical evidence, student failure is attributed to the lack of required mental structures, construction of Objects through the mechanism of encapsulation being the most difficult stage to achieve (Arnon et al., 2014).

According to Dubinsky et al. (1994) “An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms. Even if one begins with a very concrete group, the transition from the group to one of its quotients changes the nature of the elements and forces a student to deal with elements (e.g., cosets) that are, for her or him, undefined.” (p. 268)

In Brown et al. (1997) a genetic decomposition of the group concept is given in terms of the coordination of three schemas: set, binary operation and axiom; in fact, the first two are coordinated through the third one. The need for this coordination is pretty obvious since literature reports that there are students who think about a group as a set without being aware of the role that the binary operation plays and this has consequences on their subsequent understanding of Group Theory concepts (Dubinsky et al., 1994; Iannone & Nardi, 2002). As Findell (2001) notes, “The operation gives the group its structure. In other words, a group without its operation is merely a formless collection of elements” (p. 131). The failed coordination is typically evident in affirmative answers to questions of the type “Is Z_3 is a subgroup of Z_6?” (Hazzan & Leron, 1996) or when students claim “that the identity in Z_3 is inherited from Z_6” (Findell, 2001, p. 152) or even that associativity in Z_6 is inherited from Z (Findell, 2001), or still, that any subset of a group inherits closure (Fukawa-Connelly, 2007). Furthermore, Iannone and Nardi (2002) observed that students interpreted the group axioms as being properties of the group’s elements rather than those of the binary operation.

In order to illustrate the mental structures of Action, Process and Object, we can consider the example of the coset concept. A student with an Action conception can form cosets of familiar groups, such as Z_{18} but would have difficulty of doing it for groups such as D_3. As the following extract illustrates, in forming the cosets of $H = < 3 >$ the student needs to write down explicitly the steps in performing calculations in an algorithmic manner, and may be confused about whether in forming the cosets of the form aH, a runs through H (subgroup) or G (the group):

> **Cal:** Well, the number in front is what you add to each element inside the set. So zero added to these six elements would keep the same six. One [the number] added to each, which is in the first column, would give you the 1, 4, 7, 10 and then you add 2 these first the H which is zero through 6, 9, 12, 15. Then you add 2 to each and you get 2,5,8,11,14 and 17. (Dubinsky et al., 1994, p. 283)

With a Process conception, the student can think about forming the cosets without having to actually write them down and may begin to observe patterns, in forming sets of the form, for example, $1 + < 3 >$ in Z_{18}, and can decide when to stop:

> **Lon**: Okay, yeah. I should have said that $3 + H$, of course, which is a coset in its own right, is equal to the coset $0 + H$ because you get the same members as if you had added 0. Same goes with 4, 5, 6 and so on. (Dubinsky et al., 1994, p. 284)

When the Process of coset formation is encapsulated into an Object, actions can be performed on it such as multiplying them, forming sets using them as elements and preparing operation tables using labels for them. When necessary, the student can *de-encapsulate* the Object to go back to the underlying Process:

> **Jocelyn**: It's uh, for subgroups you can pick representatives and just multiply them and then your answer will be the coset that contains the product.
>
> …
>
> **Interviewer**: Do you remember what the original definition is?
>
> **Jocelyn**: Uh, I think we had to go through and multiply every single element in the first coset by every single element in the second coset. (Asiala et al., 1997, p. 256)

In the teaching approach related to APOS theory (see Dubinsky & Leron, 1994) students are expected to produce computer code with the intention to help them construct the group concept as a generic object. Students are introduced to the group notion through a computer code called "name_group" that makes use of several functions that students themselves wrote. The following quote describes the general approach of the course:

> From early on in the course, even before the concept of group was mentioned, students were working with sets which were closed under binary operations, with closed subalgebraic structures, and with group theoretic concepts. By the time groups were formally introduced, the students had already worked with a variety of examples and group theoretic concepts. The preferred way of introducing a topic was to have students explore examples relating to the topic before any mention of the topic. In this way, we prepared very fertile ground in which to plant some mathematical seeds. (Smith, n.d., p.6).

About defining concepts in an Abstract Algebra course, Leron and Dubinsky (1995) state the following:

> When the students eventually come to learn the "official", general, abstract, formal version, this is perceived by them not as totally strange and prohibitive

> (as we believe is the case in standard lectures, where such abstractions are presented without any experiential basis), but as an elaboration of their previous experience. In popular terms we may say that the activities provide an initial intuitive familiarity with the topic to be learned. In more psychological terms (supported by an elaborate theoretical framework and research), we may say that the activities help the students to "construct" the mental processes, objects and relations necessary for a meaningful understanding of the topic. (p. 231)

According to APOS theory, a Schema is considered to be thematized when it is thought of as a total entity on which actions can be applied. For example, students in the study by Brown et al. (1997) were asked to work with the following question:

> Let $(G,*)$ be an abelian group, t a fixed element of G, and define the binary operation $\Diamond$ by
>
> $$x \Diamond y = x * y * t^{-1}, x, y \in G$$
>
> Prove or disprove that $(G,\Diamond)$ is a group. (p. 237)

The ability of students being able to "think about a generic group situation and distinguish among several instantiations-especially when these instantiations have something in common, such as the underlying set- and coordinate the application of these instantiations in order to compare them" (p. 226) was interpreted as the construction of their group Schema as an Object.

Findell's (2001) research also shows the difficulty that students have with defining properties as opposed to defining sets. For example, a student, when asked to give the definition of an identity element, wrote the following response:

> There is an identity element for the group so that every element in G, when multiplied by this identity element, e, will give you back the original element: $\{x \in G \mid xe = x\}$. (p. 170)

Although she uses quantifiers essentially in a correct manner, this student ends up defining the set of those elements that operated on the right with e, stay the same. Findell also reports that students seem to have more difficulty with the definition of the identity element than that of the inverse element.

3. Definitions and examples

> Fundamental definitions do not arise at the start but at the end of the exploration, because in order to define a thing you must know what it is and what it is good for (Freudenthal, 1973, p. 107).
>
> Unfortunately, neither Freudenthal, nor anyone else, has shown us how this transition - from exploration to formal presentation - can be achieved (Gardiner, 1995, p. 254).

Durand-Guerrier et al. (2015) mention that they would like to answer the question of in what ways definitions and examples of algebraic structures such as groups, rings, fields and vector spaces as well as of algebraic objects and notions such as neutral element, invertibility, irreducibility, equivalence relation and Euclidian division form a pre-requisite for further study of abstract structures, with the aim of developing an abstract theory. Definitions are seen as "labels put on bottles to be filled with classes of concrete examples that are come across over the course of mathematical activities"; identifying the common structure of these examples in each category would help the realization of an abstract theory (p. 103).

What is meant by "definition" consequently affects the way it is employed in research. It can refer to citing definitions, creating definitions, reinventing definitions, using definitions or consulting definitions, among other activities. This in turn influences the way with which it forms a prerequisite for Modern Algebra learning. If conceptual understanding of the underlying notions is the focus, then the nature and role of the definitions in mathematical activity enter into play.

If the concern in a study is about observing the way students make use of a definition when they search for examples of the concept being defined, one can for example provide the students with a definition that they have never seen before, which might be of an invented concept, and ask them to give examples of it. This way, methodologically speaking, the problem with some students not recalling a definition could be avoided. Also, and not less important from a methodological point of view, the risk of students relying on their memories both in case of trying to remember a definition and familiar examples seen in class or in the textbook, would be reduced. This way we could get a sense of the elements that they work with in generating examples. However, there are other considerations as the following research works illustrate.

Work done by Barbara Edwards (1997; Edwards & Ward, 2004, 2008) about the status, role and use of definitions by students of analysis and Abstract Algebra shows that students may fail to conceive the definitions as being *stipulated*, as opposed to *lexical* or *extracted*. This contrasts completely with mathematicians' viewpoint on definitions and can be the cause of serious difficulties when it comes to understanding concepts that are being studied. Extracted definitions are established based on observations while in mathematics, definitions are created by attributing a list of properties to the concept to be defined and are "imposed on the reader by decree" (Wells, 2016). This also has to do with the statement made by Durand-Guerrier et al. (2015) about definitions not having a truth value; although logically true, it seems that for some students this is not so. In fact extracted definitions do have truth values since they report usage; mathematical definitions, on the other hand, being stipulated, create or improve usage (Edwards & Ward, 2004; 2008).

Research shows that reciting a definition correctly does not imply an understanding of the concept being defined nor its application in proofs or usage in problem solving (e.g. Rasslan & Vinner, 1988). In some studies, like in Durand-Guerrier et al. (2015), students were asked to recall the definitions and afterwards solve problems using them. In this approach if a student could not remember a particular definition, he or she could not tackle the problems. In other studies the focus was on the ability to apply definitions, and therefore students were provided with the necessary formulations so that they could start working on the problems; this way their success in solving them would not be limited with the availability of those definitions (Edwards, 1997). In this spirit, Edwards and Ward (2004) worked with students in an Abstract Algebra course who had not studied the definitions that were provided to them during the interviews; they were forced to pay attention to how the concepts were formulated and could not rely on their memory.

Edwards and Ward (2008) claim that there may be two reasons why students do not apply a given definition correctly. One reason might be that the student's understanding of the content of a particular definition may be incorrect. The second reason is that the student's understanding of the characteristics of a mathematical definition in general may be incorrect. The results of these studies suggested that some students do not use the definitions appropriately in solving tasks, even when they are available to them. For example one student, working on a problem involving cosets, avoided referring to the definition and instead tried to remember how she had solved similar tasks before and at times manifested a belief that mathematical definitions are extracted as opposed to being stipulated. This might be understandable, from the point of view of the students' experiences; to them, a body of mathematical knowledge is presented as ready-made, and they perceive it as extracted (Edwards & Ward, 2004). In other words, the way mathematics is taught, does not allow them to live a mathematician's experience from the beginning to the end.

Edwards and Ward (2004) discovered that even in situations for which they conjectured there would be no other way but to rely on the mathematical definition to complete a task, for some students this was not the case. Participants in this study had worked with the coset formation previously; during the interview they were provided with the definition of coset multiplication and asked to perform some. To the authors' surprise, instead of using the definition that was available to them, some students recalled their FOIL (First-Outer-Inner-Last) method that is used for multiplying expressions such as $(a + b)(c + d)$, or suggested the union of two sets as possible answers.

Edwards and Ward (2008) discuss the role of defining activity as a possibility to engage students in thinking about the role that definitions play in mathematics, in the sense that students create their own definitions so that they participate in authentic mathematical experiences. However they warn against this kind of activities, especially if they are being employed in the case of existing mathematical concepts, since they might convert into “games that are won if the student can guess what the teacher is thinking” (p. 230), hence reinforcing the idea that mathematical definitions are extracted. In order for this kind of activity to work, students should be allowed to work their way through, until they discover the logical consequences of the definitions that they formulated, including the “unintended consequences” (p. 230). Only after this phase has been concluded should the students compare their definitions to the ones actually in use (Edwards and Ward, 2008).

Although different in approach, we also mention the work of Zandieh and Rasmussen (2010) for whom *defining* involves activities such as proposing conjectured definitions, testing them through examples created for this purpose and negotiating them as well as trying to demonstrate whether the definitions do the job that they are supposed to do. In doing this, both aspects of formulating the definition and the generation of meaning are given importance (Larsen & Zandieh, 2005). Larsen (2009) reports on a project in which students were involved in intensive defining activity as part of a developmental research project (Gravemeijer, 1998). In this approach focus was placed on the guided reinvention of the formal concepts of Abstract Algebra by students, starting with what the students already bring with them, in terms of informal knowledge and strategies, based on Freudenthal’s (1973) ideas about avoiding a teaching approach for teaching group theory where first a definition is given and then examples and other results will follow.

In Larsen’s (2013) teaching design students worked in pairs on the symmetries of an equilateral triangle in order to formulate a definition for the group concept, by first identifying the rules, then reducing them to a minimum, and then working on other examples with a group structure and identifying the invariant characteristics across these examples. Some axioms needed more prompting to be considered explicitly, such as the ones involving the inverse and associativity. Students then worked on proving theorems and went on to reinventing related concepts such as isomorphism and quotient group, with the guidance of the instructor. “[T]he notion of an abstract group emerges with the reinvention of the isomorphism concept in that isomorphism makes it possible for different groups (e.g., the symmetries of an equilateral triangle and the permutations of a set of three elements) to be seen as instances of the same abstract group” (p. 721). Students also worked on inventing their own notation systems. This approach may not be too practical to be applied in a whole Abstract Algebra course and research is necessary to determine what kind of structure sense students might develop as a result of it, however it might provide the students with worthwhile experiences about the defining process. Larsen (2013) notes that Dubinsky et al.’s (1997) warning about the difficulty of abstracting the properties of the general group concept from specific examples was held true in this experiment.

Fusaro Pinto and Tall (1999) present two ways definitions are employed in mathematics. The first one, *giving meaning* (to a definition), occurs when one uses previously built concept images in order to understand a definition, including examples and visualization. *Extracting meaning* (from a definition), on the other hand, refers to working with the definition by means of deductive reasoning. One might think that these two modes of usage would be required in different kinds of mathematical tasks, however, interestingly, Fusaro Pinto and Tall identified students who preferred one or the other approach when working on one task.

Bills and Tall (1998) state that "A (mathematical) definition or theorem is said to be formally operable for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument" (p. 104). About the relationship between a definition and examples, they argue the following:

> If a student is focusing mainly on the essential properties in the definition then, in meeting new examples, there is the possibility of focusing only on these essentials, thus greatly reducing the cognitive strain. A more diffuse view of the possibilities means that successive examples may have a variety of extra detail that can cloud the issue. The former approach has prior focus on the "intersection" of the properties of the examples, the latter must sort out the important essentials from the "union" of the examples with their subtle irrelevancies that can lead to cognitive overload. (p. 111)

They add that "the struggle to make definitions operable can mean that some students meet concepts at a stage when the cognitive demands are too great for them to succeed" (p. 104). Once again, adopting the viewpoint of APOS Theory, we can say that without the mental structures that prepare students to understand the concept appearing in a definition, the pure statement of it will be of very little use to students, even if they can cite it correctly.

According to Zazkis and Leikin (2007)

> examples generated by participants – if solicited in a certain way – mirror their conceptions of mathematical objects involved in an example generation task, their pedagogical repertoire, their difficulties and possible inadequacies in their perceptions. However, there is a need for explicit criteria for evaluating examples generated by participants. (p. 15)

Starting from the assumption that "to understand mathematics means, among other things, to be familiar with conventional example spaces" (Watson and Mason, 2005, cited in Zazkis & Leikin, 2007, p. 21), they offer a framework in which emphasis is placed on three characteristics of examples generated by students: accessibility and correctness; richness; and generality. Conventional examples refer to the ones that mathematicians and the mathematical community accept; they usually form part of the curriculum and are privileged by instructors in their intent to enculture students.

Zazkis and Leikin suggest that studying the relationship between the personal and conventional example spaces of students through their framework might lead to an understanding of participants' mathematical knowledge and its characteristics. Later Zazkis and Leikin (2008) distinguish between two kinds of conventional example spaces: expert example spaces as revealed by mathematicians through their variety and richness, and instructional ones as promoted by textbooks and instructors.

Dahlberg and Housman (1997) in trying to get a sense of how undergraduates initially understand a notion, provided the students with written definitions of concepts new for them and interviewed them about their comprehension by means of questions and requests for explanations, examples or solution of tasks. The definition that was provided was that of a fine function, defined as a function that has a root at every integer point. Initially students were given the opportunity to generate their own strategies to try to understand the given definition, and after some time, depending on what the students chose to do, it was complemented by the interviewer's requests. There were four learning strategies observed: example generation, reformulation, decomposition and synthesis, and memorization. The authors found that the initial level of sophistication of understanding among the participants was highest for example generators and decreased in that order. Although the definition involved in this study was not that of an abstract algebraic structure and research can show whether the same conclusion can be reached in this case as well, user-generated examples and reflection on them as a didactical strategy might be a promising approach in helping students comprehend a new concept. However the authors warn that question design and instructor's pedagogical strategies play an important role in reaching this outcome. Iannone et al. (2011) on the other hand signal that there are not clear indications that example generation helps students in their understanding of mathematical concepts in general and with proof strategies in particular. They suggest that the quality of the way with which students engage in example generation tasks can play a significant role on what they gain from it.

There are several studies on different types and uses of examples in mathematics education literature, including at the university level. Antonini (2006) identified three types of strategies that graduate students used when generating examples: trial and error, transformation, and analysis. In this study all the tasks involved concepts already known to participants, with uncommon properties imposed on them.

Weber et al. (2008) consider three strategies that might help students use examples in furthering their understanding of mathematical concepts: "(1) by presenting examples, (2) by helping students generate examples, and (3) by asking students reason about given examples" (p. 247). They further suggest a type of task in which students are asked to provide examples of a concept that is being restricted more each time by imposing other constraints on it, in which they demand the students to make sure that no example generated for one item should satisfy the next one. After being introduced to the notion of convergent sequence and producing an example of it, students try to generate examples for a convergent non-monotonic sequence, a convergent sequence that does not approach its limit more with each term, a convergent sequence that reaches its limit and a convergent sequence whose formula, when thought of as a function, would not be continuous. If we try to employ this type of task in group theory, we might ask the students to give examples of a subgroup, a normal subgroup and an abelian subgroup of a group, for example. Or, to the tasks suggested by Durand Guerrier et al. (2015) in relation with the definition of a spanning set in Linear Algebra, we can add an item adding the condition that the spanning set be independent, hence working with the idea of a basis. Furthermore, if the students are asked first to give a spanning set that should not satisfy the conditions of the second item, i.e. linear independence, one can have the opportunity to distinguish between these two concepts, spanning set and basis, a confusion reported in the literature (Nardi, 1997).

It is also reported that for students it is easier to check whether a given object satisfies a definition or not, than to provide examples of a concept (Kú et al., 2008). This might be explained in terms of the mental constructions involved in each kind of activity. Checking properties could be done in an algorithmic fashion using Actions, whereas providing examples would require at least a Process conception. However, even when checking properties, there might be some conditions that are systematically overlooked by students. Kú et al. (2008) observe that when checking whether a given set is a spanning set for a vector space, students tend to ignore the condition that the elements of the set should belong to the space and they focus only on the generating part. This phenomenon is also reported in Durand-Guerrier et al. (2015).

Bogomolny (2006) reports about some students whose first reaction, when asked to give an example of a linear transformation, was to recall and cite the definition of this concept, and only based on that definition, search for an example. Even when their definitions mentioned the two linearity properties, these students did not seem to have a clear understanding of what the properties meant and this prevented them from giving correct or complete examples. This indicates, from the viewpoint of APOS Theory, that without having developed the necessary mental structures, it will be of little use for the student recalling the definition of a concept.

Roa-Fuentes and Oktaç (2010) describe two possible ways to construct the linear transformation concept, both of which involve coordination of the processes of the linearity properties. They (Roa-Fuentes & Oktaç, 2012) report about a student who, although correctly stated the definition of a linear transformation, insistently used only one property when trying to decide whether a given transformation was linear or not. A similar phenomenon was observed by Bogomolny (2006), as well. This kind of results is a clear indication of the gap between a mathematical statement and a cognitive understanding of it.

4. Definitions and proofs

The use of definitions in proofs merits some discussion as well. The type of definition and the way with which it is employed in proof-making are important factors in determining student success and understanding.

Fusaro Pinto and Tall (1999) consider that the interplay between definitions and deductions is a two-way interaction since "[t]o truly understand the nature of a definition requires the use of deductions to construct its implications" (p. 66). Weber (2002) classifies proofs into four categories: proofs that convince, proofs that explain (these two as reported in Hanna, 1990 and Hersh, 1993, cited in Weber, 2002), proofs that justify the use of a definition or axiomatic structure, and proofs that illustrate technique. Each kind serves a related didactic purpose in the classroom and in students' mathematical preparation. Proofs that justify the use of a definition or axiomatic structure are usually employed after a new axiomatic structure is presented to students. And as opposed to proofs that convince or explain, they place the doubt on the logical progression of the proof whereas the result or theorem to be proved is generally obvious and not questioned by students. In a way the proposed axiomatic structure is being tested on results that are already known in order to show that it works. He gives the example of proving why two plus two equals four using Peano's system of arithmetic. He adds that this type of proofs tend to be very rigorous. Using them in teaching can provide the students with a meta-level experience in terms of the usefulness and purpose of axiomatic systems.

Weber (2004) also talks about different strategies that students use in trying to prove statements. One of these, namely syntactic proof productions, involve "manipulating correctly stated definitions and other relevant facts in a logically permissible way" (p. 428) and may be driven by an algorithm. In this kind of proof semantic meaning does not play an important role and rather than being a source for understanding, definitions serve as a first statement to be used in a chain of deduction and may not contribute much to comprehension of the meaning of the fact that is being proved.

The decision of whether to use definitions or other properties in order to prove a certain statement is also a relevant feature of student understanding of mathematical structure. Weber and Alcock (2004) observed several students who, in trying to establish whether different pairs of groups are isomorphic, tried to use the definition of isomorphic groups without success. For example, when comparing $\boldsymbol{Z}$ and $\boldsymbol{Q}$, they did not think about using the fact that $\boldsymbol{Z}$ is cyclic and $\boldsymbol{Q}$ is not.

5. Structure sense in Abstract Algebra

As Simpson and Stehlíková (2006) note, Abstract Algebra textbooks and courses usually follow one of two approaches: either the definitions of concepts are provided first, with the intention that students would see examples as different instantiations of these general definitions, or through the study of examples they aim to arrive at generalizations. The first route implies working at a higher abstraction level from the beginning, while the second one is seen as more pedagogical in terms of facilitating student understanding (Skemp, 1971). In terms of APOS Theory, understanding an example as a mathematical structure requires constructing it as an Object, around which a Schema can be built. For example a group of permutations constructed as Object opens the way to examining its properties and understanding the underlying structure as a group (Simpson and Stehlíková, 2006). These authors suggest that when this approach is chosen, care should be taken so that not only the particular example is investigated, but this exploration should lead to the identification of relationships between objects and operations such as associativity and inverses.

Simpson and Stehlíková (2006) identify the following steps as shifts that students go through when passing from working with examples to thinking abstractly about the mathematical structure:

1. Seeing the elements in the set as objects upon which the operations act (which may involve a process-object shift).
2. Attending to the interrelationships between elements in the set which are consequences of the operations.
3. Seeing the signs used by the teacher in defining the abstract structure as abstractions of the objects and operations, and seeing the names of the relationships amongst signs as the names for the relationships amongst the objects and operations.
4. Seeing other sets and operations as *examples* of the general structure and as *prototypical* of the general structure.
5. Using the formal system of symbols and definitional properties to derive consequences and seeing that the properties inherent in the theorems are properties of all examples. (p. 352)

The first stage implies understanding what the objects are and how the operations work, which is far from being trivial (Simpson and Stehlíková, 2006). According to the authors this sequence of steps is reminiscent of the schema development that passes through the intra-, inter- and trans- levels (Piaget & Garcia, 1989). They add that it is difficult for these shifts to occur spontaneously, hence the need for instructional strategies to help students advance through the steps required for structural understanding in Abstract Algebra.

> What seems important in the development of an examples-to-generality pedagogy is not the free-for-all of unguided discovery, but an emphasis on the guidance of *joint attention*: on teacher and learner making sense of structures together, with the teacher able to explicitly guide attention to, first, those aspects of the structure which will be the basis of later abstraction and, then, to the links between the formal and general with the specific example. (Simpson and Stehlíková, 2006, p. 368)

Structure sense can be described depending on the level of studies involved. For example it is not the same thing studying operations at the high school level or at the university. At the university level, for the case of binary operations in Abstract Algebra, Novotná et al. (2006) established two main stages for the development of structure sense, that correspond to the first two steps that Simpson and Sthelíková (2006) identified as mentioned before; the stages are further divided into substages:

> **SSE:** Structure sense as applied to elements of sets and the notion of binary operation
>
> A student is said to display structure sense if he/she can:
>
> (SSE-1) Recognise a binary operation in familiar structures.
>
> (SSE-2) See elements of the set as objects to be manipulated / understand the closure property.
>
> (SSE-3) Recognize a binary operation in "non-familiar" structures.
>
> (SSE-4) See similarities and differences of the forms of defining the operations (formula, table, other).
>
> **SSP:** Structure sense as applied to properties of binary operations
>
> A student is said to display structure sense if he/she can:
>
> (SSP-1) Understand ID in terms of its definition (abstractly).
>
> (SSP-2) See the relationship between ID and IN: ID → IN.
>
> (SSP-3) Use one property for another: C → ID, C → IN, C → A.
>
> (SSP-4) Keep the quality and order of quantifiers.
>
> (SSP-5) Apply the knowledge of ID and IN spontaneously.
>
> Abbreviations ID, IN, C, A stand for identity, inverse, commutative property, associative property. (pp. 250-251)

Even these first two stages in the path to constructing mathematical structures is complicated for students. About the recognition of the elements of a set and a binary operation defined on them, Parraguez and Oktaç (2010) observed that some students, although given explicitly the operations on a vector space, when trying to decide whether a set of vectors are linearly independent, set up the equations with the usual operations of addition and multiplication. Aguilar and Oktaç (2004) report about a group of teachers who, working on a cryptography problem in the context of modular arithmetic, use the usual operations of addition and multiplication and behave as if the elements of a set Z_n are rational numbers.

The identity element for many students is identified as being the zero-element, without paying attention to its properties in terms of the operation involved. So, some students will declare that a structure does not have an identity element if there is no zero in the set (Stehlíková, 2004, cited in Novotná et al., 2006). However the authors also note that "[t]he image of 0 as the additive identity does not always have to function as an obstacle. For some students, it serves as a generic model of additive identity and they can reconstruct its properties in ordinary arithmetic and use them as a tool for finding out the identity in another structure" (Stehlíková, 2004, cited in Novotná et al., 2006, pp. 255-256).

Novotná et al. (2006) identify at least three paths in coming to understand a structure (V stands for a property or an object, index A for a familiar structure, index B for an unfamiliar structure and D for a definition):

$$V_A \xrightarrow{abstraction} D \xrightarrow{construction} V_B$$

$$V_A \xrightarrow{analogy} V_B \xrightarrow{abstraction} D$$

$$D \xrightarrow{construction} V_A, V_B$$

In the first path properties are extracted from a familiar structure to form the basis of a definition, from which the abstract concept is constructed in a general context. In the second path extracted properties from a familiar structure lead to its generalization and then to a definition. The third one corresponds to the construction of a concept through logical deduction from its definition (Harel and Tall, 1989, cited in Novotná et al., 2006). As the authors note, this model serves only for the construction of the concept of binary operation; a model for a group, which involves a binary operation, a set and their coordination through axioms (Dubinsky et al., 1994) would have to be much more complex.

6. Conclusion

The learning of Abstract Algebra is a complex process, hence pedagogical strategies should take into account research results in trying to address the identified difficulties. Research design on the other hand plays an important role on the kinds of information that we can gather. Asking students to recall facts or definitions may provide access into their repertoire whereas resolving novel situations can give information about their mental structures. For example given a set asking for an operation to create a group structure would necessarily invoke students' conceptions of Process and Object. Given a set and a binary operation, defining the other operation to give rise to a vector space structure would shed light on the coordination of the two operations or its absence in the mind of the student (Parraguez and Oktaç, 2010); this kind of information would be difficult to obtain by asking to verify whether a given set and two operations satisfy the vector space axioms.

There is no doubt for the need to perform more studies on Abstract Algebra learning, with the hope that they in turn would help give rise to pedagogical suggestions in facilitating the transition to upper level courses where abstract structures are studied.

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