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THEORETICAL DEVELOPMENTS IN MATHEMATICS EDUCATION RESEARCH: ENGLISH AND FRENCH PERSPECTIVES IN CONTRAST

Abstract. This article traces the development of theoretical perspectives in the English and French mathematics education research cultures from the 1960 and 70s to the present. The main parts of the article are the separate accounts of development in the two domains. The two areas are presented separately since they are very different both in terms of what is in focus at different times and in terms of the theories originated, developed or appropriated. The place of *a priori* mathematical analysis (i.e. analysis of the mathematics to be taught, prior to teaching) seems a key difference, beyond the institutional and cultural differences. The final part of the paper draws attention to key areas of difference between the two domains and suggests key questions and issues in which there is common ground albeit addressed from the differing perspectives and cultures.

Keywords. Mathematics education, constructivism, socioculturalism, activity theory, didactical theories.

Résumé. Développements des recherches sur l'enseignement et l'apprentissage des mathématiques – regards contrastés sur les cas anglais et français. Cet article retrace le développement des perspectives théoriques des chercheurs concernés par les questions d'éducation mathématique en Angleterre (et dans les pays de tradition anglais) et d'enseignement et d'apprentissage des mathématiques en France (et dans les pays de tradition francophone), des années 60-70 à maintenant. C'est une présentation en deux volets successifs qui occupe la plus grande partie de l'article, tant les différences sont importantes – concernant aussi bien les origines des recherches que leurs fondements théoriques. La place des analyses mathématiques semble constituer une différence majeure, par delà les différences institutionnelles et culturelles. C'est ce que reprend la dernière partie de l'article, dégageant les principales orientations de chaque pays en les mettant en regard, et présentant des questions majeures communes qui restent néanmoins posées aux deux communautés.

Mots-clés. Didactique des mathématiques, constructivisme, socioconstructivisme, théorie de l'activité.

Introduction

This article traces the development of theoretical perspectives in the English and French mathematics education research cultures from the 1960s and 70s to the present. Initially, we deal with the two areas separately since they are very different

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both in terms of what is in focus at different times and in terms of the theories originated, developed or appropriated. Necessarily there is a strong historical dimension in each as theories are related to events, educational trends and developments across 50-60 years. The main parts of the article are the separate accounts of development in the two domains. These are necessarily lengthy in order both to cover the range of theories and address associated educational structures and issues. The final part of the paper draws attention to key areas of difference between the two domains and suggests key questions and issues in which there is common ground albeit addressed from the differing perspectives and cultures. We are aware that there may appear initially some inconsistency between the two parts, but it reveals a great difference between the two developments – mainly because the French one was developed in contrast to the education sciences, with the mathematical content analysis up front, although some concepts are borrowed or shared. In contrast the English development has to be understood as a part of these sciences, even where specific mathematical content is taken into account. These differences may not be reduced into a uniform presentation.

Two of the authors are associated with each of the two domains, Aline Robert and Eric Roditi with the French domain and Stephen Lerman and Barbara Jaworski with the English domain. We speak of the "English" domain in general rather than the "United Kingdom" domain to emphasize that international theoretical trends in English-speaking countries have influenced the domain, rather than developments only in the United Kingdom. However, the educational perspectives historically pertain mainly to the United Kingdom. In the French case, the developments discussed arise first within France and relate to the history of educational development in France. It will also be evident to readers that the English history evidences a wide range of theories whereas the French history is much more focused on a few overlapping theories.

1. Theoretical developments in English mathematics education research (Stephen Lerman and Barbara Jaworski)

1.1. Early influences

Research in mathematics education in the United Kingdom has a history dating back to the late 1800s. We can point to the founding of the Association for the Improvement of Geometry Teaching in 1871, renamed as The Mathematical Association (MA) in 1894, as perhaps the beginnings of the field. It has had many eminent mathematicians as its president, including A.N. Whitehead, co-author with Bertrand Russell of *Principia Mathematica,* from 1915 to 1916. The MA was dominated by secondary school teachers, mainly from private and grammar schools. A breakaway group, led by Caleb Gattegno (an influential figure in education at that time) founded The Association for Teaching Aids in Mathematics

(ATAM) in 1952 to focus on primary as well as secondary mathematics teaching. A decision to change the name of the association to the Association of Teachers of Mathematics (ATM) was initiated at the 1962 AGM and took effect in June 1962. As it says on their website, “An early aim of the Association was that all children should learn mathematics through lively and interesting experiences”, an egalitarian direction that has formed a feature of research in mathematics education in the United Kingdom since then. Both associations, the MA and the ATM, have remained active in the field of research, holding conferences and being productive in publications, and incorporating teachers and researchers as members.

As we indicate in the next section, we can perhaps take the early years of the ATM as the beginning of the modern era in mathematics education research in the United Kingdom. Gattegno’s work, which viewed working mathematically as a central part of all human functioning, was and remains a huge influence on teachers of mathematics. In these early years a strong influence also came from the work of Jean Piaget, particularly his clinical interviews and stages of intellectual development (Gruber and Vonech 1977). These influenced a Government report on primary education, whose committee was headed by Lady Plowden, published in 1967. The Plowden report led to a revolution in primary education, introducing the concept of *child centeredness* into the language of teaching and curriculum, highly consistent with the thinking within ATM. These theoretical beginnings can be seen as a forerunner of the practically focused and wide-ranging theoretical orientations discussed below.

1.2. Wide-ranging orientations

In our brief survey of theoretical orientations in United Kingdom (henceforth UK) mathematics education research since the 1950s and 1960s which follow, we suggest that a range of perspectives have been drawn upon by researchers, evolving and developing over those years. We have set out a broad timeline (see Figure 1) and will expand on the developments below. It is perhaps typical of the rather eclectic and practically focused approach of British intellectual thought, possibly even across all the English-speaking world, that there should be a range of orientations, rather than a strong and unified set of theories common to nearly all researchers as is the case in France.

Furthermore, we would emphasize that there is no sense in which we can speak of a ‘progression’ across the decades. It is a phenomenon of the social sciences in general and education in particular that new languages of research emerge and sit alongside existing ones, a phenomenon that the UK sociologist of education Basil Bernstein called a horizontal knowledge structure (Bernstein 2000). Thus, Piaget’s child development theories did not replace behaviourism, nor did the emergence of Vygotsky’s work in mathematics education in the late 1980s lead to a move away from Piagetian theories. More recently, postmodern critiques and methodologies have been developed that, once again, sit alongside existing theories (see Lerman 2000, for a more developed account) developing a language of research within their own set of theoretical structures and over time continue to proliferate.

We might suggest that this proliferation is at least in part responsible for the lack of a progression. What one researcher or group might consider progress may well be criticized by another group with a different orientation. We leave further discussion on these matters to a later part of this paper where we contrast and compare the English with the French research traditions in mathematics education. In regard to references, at an earlier stage of the writing we began to reach a bibliography that covered ten pages. This is not possible for a journal paper. We have decided, therefore, to restrict the references severely, and will list just those that we consider to be essential. In many places the names of scholars associated with developments in research will be mentioned, giving readers leads to further references.

1.3. Beginnings of modern developments in English research and influences in the United Kingdom

Piaget's developmental psychology and practical orientations in United Kingdom research

The *British Society for Research into Learning Mathematics* (BSRLM, originally BSPLM where P means Psychology) was founded in 1976, the same year as the International Group for the Psychology of Mathematics Education (PME), by Richard Skemp, Celia Hoyles, Kath Hart, Alan Bell, Margaret Brown, David Tall, and others. The early field was extremely influenced by the *Psychology* of Mathematics Education and this influence continues to an extent to the present day.

This psychological tradition in empirical research in the UK derives strongly from the work of Piaget, himself both a theoretician and empirical researcher in psychological traditions. A leading exemplar of this orientation, in the late 1970s, was the "Concepts in Secondary Mathematics and Science" (CSMS) project, led by Kath Hart at the London University Chelsea College. Based on hierarchies of biological development the CSMS team surveyed students across the UK and developed levels of progression across a range of topics of school mathematics (e.g. Hart 1981). The findings of this study have permeated teacher education courses and influenced teaching and curricula over 20-30 years. Also influential has been Mellin-Olsen and Skemp's distinction between forms of understanding which they classified as *instrumental* and *relational* (Skemp 1971) : the *relational* being understanding in which concepts and their use are understood as a basis for mathematical activity, whereas *instrumental* understanding implied a use of rules or procedures, often without a conceptual underpinning. This distinction was seen by Skemp as an essential extension of Piaget's work on understanding. The influence of Piaget can also be seen in extensive work on diagnostic assessment, on cognitive conflict and conflict discussion, much of it taking place at the Shell Centre in Nottingham and at Kings College London (which absorbed Chelsea College in the early 1980s). We can now see these areas of more local theory, within the Piagetian perspectives on intellectual development, as forming a practically rooted theoretical base for modes of classroom activity.

A philosophical turn emerged in the early to mid-1980s, developed, in particular, by Paul Ernest and Stephen Lerman, with Ernest continuing that body of work until today. That work saw itself drawing particularly on Imre Lakatos's fallibilistic philosophy of mathematics (Lakatos 1976) and was strongly associated with the Radical Constructivist tradition, based on Piaget's theoretical ideas on learning that was growing in strength in the USA through pioneering work of Von Glasersfeld, Cobb, Confrey, Steffe and others (see Glasersfeld 1991; Cobb and Steffe 1983). The UK community was not swept along with Radical Constructivism to the same extent as USA colleagues; however, concepts from constructivism and radical constructivism became useful to some researchers in the UK. Nevertheless Piaget's developmental psychology was hugely influential in schools and broadly a firm theoretical background for UK mathematics education researchers. The hierarchy of knowledge in mathematical topics based on stages of intellectual development developed in the CSMS project, and the attention given to common errors and misconceptions, influenced the development of the first National Curriculum for Mathematics in the UK, in 1988.

As we have suggested above, at the roots of theory development in the UK, and influencing its diversity, is an exploratory, investigative tradition in classroom practice and its development, with teachers engaging in classroom research alongside teacher-education researchers from university education departments. Historically and significantly, this investigative tradition in teaching and learning mathematics was represented in the work of the ATM with its influential journal *Mathematics Teaching*, and annual conference including workshops for teachers and researchers to explore mathematical ideas. This activity was complemented by the early days of the Open University mathematics programme in which all mathematics students, many of whom were teachers, had to attend a summer school during which they engaged in investigative activity. In classrooms, an investigative approach to learning mathematics was encouraged through curriculum support materials such as the Kent Mathematics Project (KMP) text books, the SMILE series of work cards for students (see http://www.greatmathsteachingideas.com/smile-mathematics-resources/) and the School Mathematics Project (SMP) series of books (some of which are available here: https://www.stem.org.uk/resources/collection/283319/school-mathematics-project).

A practical tradition was established in which classroom mathematical activity developed through the work of inspired teachers and educators (such as Dick Tahta, John Mason, Eric Love). Love wrote a seminal article in the book *Mathematics, Teacher and Children* (Pimm 1988) called “Evaluating Mathematical Activity”. Jaworski’s study of investigative practices (Jaworski 1994) linked the investigative tradition in classrooms with the theory of radical constructivism. The work at the Shell Centre in Nottingham on cognitive conflict discussion (introducing conflicts into classrooms dialogue to promote accommodation of mental schemas) fitted with the exploratory ambience as did John Mason’s “*Theory of noticing*” (Mason 2002). Mason’s theory encouraged teachers to ‘notice’ aspects of their practice relating to tensions or issues in teaching/learning and to reflect on them, both *after* teaching and *in* teaching. Reflection *in* teaching could then lead to opportunity to change the action ‘in practice’ rather than in future planning. Thus inquiry within teaching practice itself was both theorized and promoted. Critiques of constructivist theory, and particularly of radical constructivism, suggest its dualistic nature - a paradox of positing an inner subject experiencing an outer world, resulting in the human subject constructing a representation of the world. Seeking to avoid this claimed dualism, the theory of *enactivism* avoids the inside-outside dilemma. Using a metaphor of « a path laid while walking » (e.g. Dawson 2008) in which “all knowing is doing and all doing is knowing” (Maturana and Varela 1987, p. 27) enactivism is essentially a non-representationalist view of cognition. In other words, our knowing is in our action and vice versa, or to quote Maturana and Varela (1992, p. 29). “Knowing is effective action, that is, operating effectively in the domain of existence of living beings”. Laurinda Brown and Alf Coles are UK scholars working with enactivism (Brown and Coles 2011). All of the work referred to above was very much in the practical tradition with research being closely associated with ‘activity’ in teaching and learning.

This practical tradition was also seen in the early days of the UK National Curriculum (introduced for the first time in 1988) which had a strand on “Learning and Doing Mathematics”. The inclusion of assessed coursework for students which was investigative in style in the national examinations at age 16 led to all schools focusing on investigations in mathematics classrooms. Attention to issues of equity and diversity grew through this practical tradition, with practices of differentiation and inclusion growing through in-service work with teachers, and in initial teacher education programmes (people such as Laurinda Brown, Anne Watson, Peter Gates). Research in teaching became important in order to conceptualize teaching beyond anecdotal practice. Through PME, research in teaching was made more public with working group publications – collections of papers from research into teaching around the world (e.g. Vicki Zack, Judy Mousley and Chris Breen; Barbara Jaworski, Terry Wood and Sandy Dawson). Since then theories of teacher knowledge and practice have extended and grown, as work by Tim Rowland, Kenneth Ruthven and others demonstrates.

Digital technology in mathematics teaching and learning.

To add to what we have written above, we need to address an important, although somewhat separate dimension of mathematics education research, that of the integration of digital technologies in the teaching and learning of mathematics. It seems fair to say that early activity in the UK drew on two important dimensions:

1) An interest in computer programming led by mathematics teachers and researchers in the MA and ATM;
2) The work of Seymour Papert at MIT, focusing on the theory of constructionism (different from constructivism in several important respects, including dualistic imputations and the importance of language and discourse).

Activity deriving from (1) was almost entirely practical rather than theoretical. It coincided with an era of technological development in which schools started to use microcomputers (e.g. the BBC micro) and started to teach Computer Studies/Science.Students were encouraged to write simple programs (in the language BASIC) and to understand the working on computers in a range of applications.

Activity deriving from (2) also involved computer programming, largely in the language LOGO, or simplified versions of it involving Turtle Geometry, as developed through the work of Papert (Papert 1980). Scholars in the UK using Papert's theoretical perspectives in researching the use of LOGO included Celia Hoyles, Richard Noss, Ronnie Goldsten and Janet Ainley. From this early work, Hoyles and Noss developed their theory of *Windows on Mathematical Meanings* which was an extension of constructionism (Noss and Hoyles 1996). Their work led to further developments within the UK in which students were encouraged to work within technological micro-worlds constructing their own computer-based models in solving mathematical problems.

In parallel with this work in the UK, and consistent with Papert's philosophy, colleagues in France were developing dynamic software to support the teaching and learning of geometry. Colette and Jean-Marie Laborde introduced the software Cabri-Geometre, which was designed to engage students in collaborative exploration of geometrical concepts (e.g. Laborde 1995). This was highly influential on geometry teaching worldwide and the forerunner of other such software (such as GeoGebra). Also in France, a theory of *Instrumental Genesis* emerged through the work of Luc Trouche and Ghislaine Gueudet, capturing relationships between the digital medium and the user of this medium in an educational context. While the impacts of this work were international, they were also significant for scholars working with computer-based media in the UK.

Research into mathematics teaching and learning in higher education

Most of the research referred to in this section above has taken place in primary and secondary education; theoretically-based research in higher education in mathematics has been less visible during these times. An exception has been research into so-called 'Advanced Mathematical Thinking', largely rooted in Piagetian or constructivist ideology and developing from the seminal book edited by David Tall (Tall 1991). David Tall has been a key figure in the field since the 1970s. Most recently he has developed a comprehensive account of human development and of teaching, built around both psychology and the nature of mathematical thinking (Tall 2013). This work is probably unique internationally, in that such a comprehensive account, which also attempts to incorporate all the substantial developments in the field, cannot be found elsewhere.

Tall's work has been particularly influential on research on university-level mathematics education (e.g. Tall 2008). Spurred by the publication of *Advanced Mathematical Thinking* (Tall 1991), a number of researchers have aimed to understand the cognitive processes involved in advanced mathematics. Particular focuses have included the construction and evaluation of mathematical proofs (e.g. Weber and Alcock 2004), students difficulties with definitions (e.g. Alcock and Simpson 2017), mathematicians' epistemic cognition (e.g. Weber, Inglis and Mejia-Ramos 2014), as well as detailed analyses of students' difficulties with particular concepts in undergraduate mathematics (e.g. Pinto and Tall 2002).

During the period from 2000, research activity at the higher education level has become more diversely theoretical. As more mathematics educators have started to study teaching and learning within the university, other theories have been used to make sense of educational practices in the UK – notably Commmunity of Practice and Community of Inquiry (Jaworski 2014) and Commognition (Nardi, Ryve, Stadler, and Viirman 2014), introduced by Anna Sfard (Sfard 2008) and focusing particularly on language and discourse in mathematical learning and teaching. We see these new theoretical directions to be influenced by moves away from Piagetian constructivism towards sociocultural perspectives on knowledge, drawing extensively on the work of Vygotsky and other theorists in sociological domains as we address in Section 1.4 below.

1.4. Sociocultural and sociological approaches

During the late 1980s, Vygotsky's cultural developmental psychology, with its intellectual roots and theory of learning and teaching, radically different from those of Piaget, became known in the UK in mathematics education, and around the world, influenced by Jerome Bruner's seminal talk in Geneva, "Celebrating Divergence : Piaget and Vygotsky", (Bruner 1997). Its knowledge and influence began to permeate thinking and practice from the mid-1990s. The notion of scaffolding, a popularized but, we would argue, also inappropriate term for the zone of proximal development, became ubiquitous in the education world, including Government documents for education. Mediation, activity theory, and the zone of proximal development (Wertsch 1991) became research foci amongst some parts of the mathematics education research community in the UK. An early example of this in the UK can be seen in the work of Simon Goodchild who analyzed 'Students' Goals' in the mathematics classroom using activity theory concepts and Jean Lave's cultural psychology (Goodchild 1995, 2001; Lave 1988).

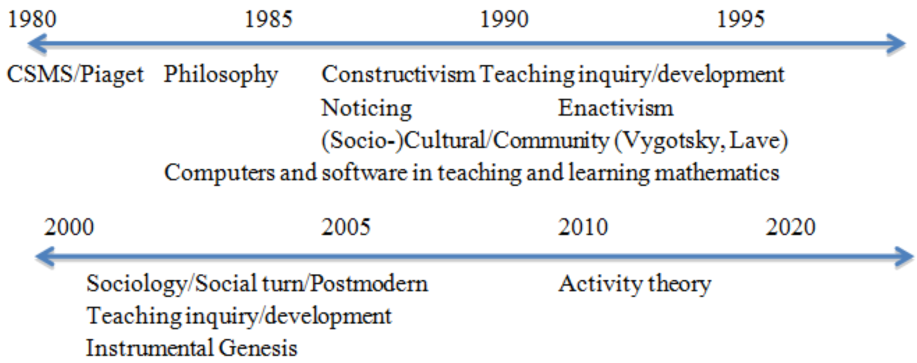

Figure 1 : Suggested timeline for theoretical development in the English World

Activity theory, either in its first generation form from Vygotsky of the mediation triangle, the second generation form from Leont'ev of activity, action and operation, or the third generation form from Engeström, has become a growing tradition of research in the UK beginning late in the 20th century (Leont'ev 1981 ; Engeström 1999). It is certainly as a consequence of these Vygotskian developments that researchers very often refer to sociocultural aspects in their research. This is often taken to mean a recognition of the need to pay attention to students' or teachers' wider social context when accounting for learning, teaching or both. Thus issues of social class are addressed: such as whether those being researched are from advantaged backgrounds or are what has been called poorly served students in those schools in lower socioeconomic settings may have greater changeover of teachers, less qualified teachers and so on. Language and culture may be discussed and examined. It is not always the case, however, that learning–teaching as a cultural-historical process feeds into research questions, design or analysis. Work is required on the part of the researcher to draw on the literature of sociocultural theory in such a way that the significance of Vygotskian theory underlies and informs the research studies. Such issues and concerns have become

the focus of the conference Mathematics Education and Society, discussed further below.
Many researchers attempt to construct a combination of Piagetian and Vygotskian ideas to inform their work. They often tend to use Vygotskian concepts for focusing on the group, class or school as a whole when constructivist learning theories, used for analyzing classroom interactions, do not provide a theoretical basis for the wider settings. For example, Potari and Jaworski (2002) used constructivist theory in elaborating the Teaching Triad through analysis of teacher-student interactions in classrooms, a *micro* analysis. However, they found it difficult to include *macro* considerations, such as social and cultural issues in lives outside the classroom. To include these factors in analysis, they turned to activity theory (Jaworski and Potari 2009). Notions such as the negotiation of meaning and knowledge, and opportunities set up by the teacher to support students' constructions of mathematical concepts are common in UK research today in mathematics education, and beyond (Lerman 2013). Group problem solving or an examination of the rules and goals of an activity are also taken to be features of what is often called social constructivist research. Constructivist, social constructivist and sociocultural theories and their differences continue to be discussed. Many of the important theoretical issues were presented and discussed in a special issue of the journal NOMAD (Volume 8 No. 3, 2000).

A development of Vygotsky's sociocultural programme, that of situated cognition and communities of practice, emerged in the 1990s, following the publication of Lave's 1988 book and her book with Wenger in 1991 (Lave 1988; Lave and Wenger 1991). Jean Lave visited the UK in 1996/7 and an influential seminar was held in Oxford, followed by the publication of a group of papers (Watson 1998) and a further reflective collection some years later (Watson and Winbourne 2008). Communities of practice (CoPs) as described by Wenger (Wenger 1998), based on concepts of participation and reification, along with identity, community, practice and meaning, seem to offer researchers a way of focusing on the group when studying teaching and learning. Learning as participation in the community is a Vygotskian idea developed through Wertsch and Lave, the latter through her studies of learning in cultural contexts in West Africa. In early studies, deriving from CoP theory, there was a danger that CoPs were seen to be everywhere: all kinds of situations in classrooms were described as communities without either examining and identifying Wenger's main components of learning.as mentioned above, and without the complex but very important ideas of legitimate peripheral participation as worked out in Lave and Wenger's book (Lave and Wenger 1991)

Inequalities

It is said that people in the UK have always been conscious of social class, and inequality has therefore formed a strong direction of study and action for many decades. This concern was expressed in ATM's goals as described above. Work in mathematics education research with a concern for equity took a major step when the first *Mathematics, Education and Society* (MES) conference was held in Nottingham in 1998, though predecessors include the two groups *Social Perspectives of Mathematics Education* (Nickson and Lerman 1992) and *Political Dimensions of Mathematics Education* (Julie, Angelis and Davis 1994). Its goal was to support the research in such perspectives in the UK and internationally, the founders Gates and Cotton (1998) arguing that the leading international research group, PME, was too restrictive in its theoretical demands on contributors to allow alternative research paradigms, the sociological, political, and sociocultural at least. If a research contribution did not refer to psychology in some way, it would not be accepted for presentation at the PME conference. That changed formally in 2005 when PME's constitution changed to allow a much wider range of theories to constitute the framework for the research, although the vast majority of research reports presented at PME remain informed by psychology. MES, however, has taken on a very substantial and important life of its own, and it continues to grow in size and influence. Its ninth conference was held in April 2017 in Greece.

Sociology

A different direction for research in mathematics education in the UK has come from sociology. Sociology of education is a well-established field, drawing mainly on Durkheim and Marx, and there are many international journals of sociology of education. Basil Bernstein (2000) has been the major influence in the UK, South Africa and many other countries. His work draws connections directly between the macro features of society, in particular power and control, and the micro issues of the relationships between teachers and learners and who has access to what forms of knowledge. Dowling, Brown, Evans, Tsatsaroni, Morgan and Lerman are just some of those whose work has been located in sociology since the 1990s. Bernstein's theoretical framework enables insights into how curriculum, schools, Government policies and social class pressures lead to maintaining privilege and denying access to success in mathematics to those from disadvantaged backgrounds. Revealing where these policies come from and how these processes take place to allow and deny access are the first steps in being able to make a difference in classrooms, though social structures of society are, of course, not available to us to change. The Marxist origin of sociology of education, and Bernstein in particular, means that there is a strong overlap with Vygotsky's work, his theoretical framework being inspired by Marxism too. Indeed Bernstein wrote the Preface to Daniel's 1993 book on Vygotsky (Bernstein 1993), indicating clearly there that although he, Bernstein, was a structuralist nevertheless his work did not align with Piaget, also a structuralist, but with Vygotsky.

Semiotics

A well-developed sub-field of research in mathematics education, both in the UK and beyond, is that of semiotics. Saussure's work provided a point of departure in the early 1990s into language and meaning by the Manchester Metropolitan University group (notably Tony Brown and Olwen McNamara), in Peircean semiotics (Adam Vile), and since then the chief UK proponents of mainstream Anglo-American linguistics have been David Pimm, Candia Morgan, Tim Rowland and more recently Richard Barwell (e.g. Pimm 1987).

1.5. Postmodern theories

A different orientation in teacher education research and also in mathematics learning in general has emerged from the poststructuralist/postmodern traditions, including Tony Brown, Heather Mendick, Margaret Walshaw and others (e.g. Brown 2011).

The forerunner of this direction is Valerie Walkerdine (1988, 1997) whose gender studies in mathematics education were informed by Foucault in particular. The move from structuralist work, such as Bernstein, to poststructuralist work has led to studies at a local level of the play of power through language. The two key features of these approaches, in the sense of aspects that have informed educational research, are the location of meanings in the local, and in the sources and effects of power.

Meanings in the local

Modernism is characterized by meta-narratives, including Marxism, religion, psychoanalysis, scientism (the notion that scientific research is value-neutral and a 'good' in itself), and capitalist values such as the free market. The break to postmodernism in cultural and social studies was marked in particular by the sense of failure of the meta-narratives to provide universal meaning. Meanings and values, it is argued, are to be found and developed at more local levels, including a recognition of multiple 'locals' of gender, race, ethnicity, religion, and social class that make up the multiplicity of social environments in which each of us moves. The turn to postmodernism points to methodology in particular and calls for ethnography to excavate meanings of students in the classroom, of student teachers in training or in school practice, of teachers in their own contexts, and other lived situations (e.g. Lather 2007).

Effects of power

Relations of power in educational contexts have always been in the consciousness of researchers in education. The high status of mathematics, in the social capital a mathematical qualification carries, in the intellectual status it seems to bestow on those successful in mathematics, and in the ubiquity of the applications of mathematics in society, perhaps singles out the contexts of mathematics education as especially implicated in power. For the most part, research narratives building on the disciplines of psychology, particularly Piagetian theories, the nature of mathematical knowledge, philosophy and others enable an analysis of the effects of power in quite limited ways. Foucault's identification of power with knowledge opened new and very fruitful dimensions for research in mathematics education (e.g. Walshaw 2004).

A new series of research meetings, the *Mathematics Education and Contemporary Theory* conferences, held in Manchester, has extended the semiotic and postmodern work nationally and internationally. Foucault's notions on power/knowledge, Derrida's deconstruction, Rorty's pragmatism and other theories are played with at that conference and in publications. Special issues of *Educational Studies in Mathematics* (ESM Vol. 80, 1/2) emerging from presentations at those meetings demonstrate the body of work developing.

Much has been done over decades in gender studies (see e.g. Burton 1991) and social class, on learning over time, and on analyzing accounts of participants. Postmodern theories have been central in these studies. In relation to these more recent developments, the introduction of postmodern and poststructuralist theories, a question to be asked is "how do they inform mathematics education?" Education can be seen as a *region* (Bernstein 2000), by which is meant that, unlike sociology, psychology, mathematics, and other fields, education has a face to theory and a face to practice. Medicine is another example of a region. Being informed by the disciplines of sociology, psychology, mathematics and others, mathematics education, a sub-field of education, seeks always to see how theories can be seen to shed light on practice, in this case the practice of teaching and learning mathematics. New theories appear and are applied in this way, as a lens on practice, seeing differently and interpreting differently. That these new ideas gain purchase in the sub-field depends on the usual 'gate-keeping' processes of journal review, research grant application, and PhD success or otherwise. It can certainly be said that these theories have provided new insights into power relations and equity issues. Just one example, Mendick's analysis (2006) of girls who do well in mathematics in school but choose not to take it on into the University entry level, the 'A' levels, is set within notions of identity formation and gender, an approach that arises out of postmodern theory.

2. Theoretical developments in French mathematics education research (Aline Robert, Eric Roditi in collaboration with Isabelle Bloch)

In this section, the French authors set out their perspective on the development of mathematics Education research in France from the 1970s to the present day in two parts (70s-80s, 80s-to the present). This research is called Didactics of Mathematics – and this is not anecdotal, as it reveals the intention of a split-up with the Education sciences. In fact, we have taken into account only research clearly identified as didactical research, even if there is in France other research streams such as psychology, sociology, and education sciences. Some of them may concern mathematics teaching or learning but without the focus on mathematical content that is a specific factor in didactical research resulting from the 1960s.

2.1. The development of research in mathematics education: the beginning and the first stage (70s-80s)

A brief reminder of the French context of the emergence of the specific research field called "Didactics of mathematics"

The institutional setting of the so-called "Modern math reform", from 1960 to 1970, brought a great need for mathematics teachers' education. At the same time the social conditions, tied to the students and other movements in 1968, gave rise to a real movement to provide education to all society levels (democratization of education), including university. Unfortunately it was followed by the beginning of the economic crisis from 1974 which changed the perspectives. It is interesting to notice that, probably according to this live context, some personalities revealed a great interest into issues in teaching mathematics (mathematicians, historians…).

The first institutional response was the creation of the network of the IREM (Research Institutes into Mathematics Education), the first three in 1969 and the others, 28 in the whole France, later. And, according to the new training needs for teachers, tied to the reform, and to the start up of this new structure (IREM), a lot of young mathematicians (recruited at university), began to train mathematics secondary teachers in the IREMs. Many pedagogic problems emerged from changes in the school curriculum and from the democratization it was expected to bring (even if results did not live up to these expectations). These mathematicians became quite naturally the first researchers in the field of didactics of mathematics, as developed by Guy Brousseau starting from the 1960s. Some older mathematicians joined them, as they had previously thought about mathematics teaching and empirically explored the new curriculum in some classes.

From research in Education to research in Didactics of mathematics.

It is important to notice that, at the same time in France, the constitution of the "Educational sciences" as an academic field was established, dominated by philosophers and sociologists at the beginning. But subject knowledge was not central to their inquiry. This orientation explains in a large part the need for another scientific approach, centered on subject-based knowledge, and, where mathematics was concerned for the didactics of mathematics. It is meaningful to notice that the university-based French researchers struggled for years to link institutionally to the Mathematics Department and not to the Educational Sciences Department.

First research and theories

In the 1970's, French research in didactics had been inspired by educational science research, referring first to Piaget (and later to Vygotsky) according to Bachelard (even if in his theory the obstacles did not concern mathematics specifically (Bachelard 2000, p. 26). But the need of a systemic analysis of mathematical knowledge – and the way it could be complemented or carried out – emerged, as a crucial tool to be able to understand *how* students learn mathematics. Here, cognitive models are limited, since they pay little attention to the subject matter. In fact, cognitive models might seem to suggest that issues related to learning relate personally to the learner and that difficulties may come from individual deficiency, rather than from the mathematics in focus.

For didacticians, the access to mathematical knowledge depends first on the epistemological analysis of mathematical objects: the specificity of mathematical knowledge is of great importance, as are also the conditions of teaching. Didactics aims at a systemic analysis of teaching and learning processes in an institutional context, and this leads to adapted theories and models. Didactic research does not deny the existence of cognitive operations within individuals, but didacticians aim at identifying the link between mathematics knowledge and, for instance, situations in which this knowledge can prove to be relevant – and then, effective to learn. This approach maintains a strong component of the specific nature of mathematical knowledge, with global and local mathematical analysis of the contents to be taught, leading to a conception of adapted learning situations, which have to be explored further. As difficulties in learning cannot come only from individuals, the complexity in the learning of mathematics recognizes that students can meet some common obstacles, which have to be explicitly studied, and to be taken into account in the elaboration of teaching situations. However, some of the obstacles might be created by the teaching itself, which needs to be avoided.

In this perspective, in the 70s, various researchers elaborated theories that are adjusted to different contexts. We can notice that some first studies focused on primary school mathematics, one reason being the desire to begin with the first development of the child, as Jean Piaget and Gérard Vergnaud did. Another reason may be that at this period, teacher training had not been currently developed at secondary level. Theories are rooted in *questions*, mathematical and professional; they have been elaborated to investigate these questions and, further, to build didactical engineering, not as an end in itself but as an experimental methodology.

The theory of conceptual fields (Vergnaud 1991) can be seen as a transition between cognitive models and didactics studies, as it is inspired by cognitive observations. However, it has a focus on mathematics (mainly at primary level) and it provides an interesting model concerning the way students can deal with a mathematical concept. This concept would be analyzed through three components:

- the collection of situations (problems) in relation to the concept;
- the operational invariant that take place into the resolution of a problem, since the concept is at stake in this problem; and
- the semiotic signs involved in the resolution.

As Vergnaud states:

> "It is a psychological theory of concepts, or better of the process of conceptualizing reality: it enables us to identify and study the continuities and discontinuities between different steps of knowledge acquisition from the point of view of their contents." (1991, p. 133)[1]

Vergnaud's model provides good analysis of the students' work and access to mathematical concepts, and in this way it has been a fruitful transition between a psychological approach and the intentions of building didactical situations for the learning and teaching of specific concepts.

From the 1960s, Guy Brousseau's ambition was both to build a broad model concerning the field of mathematics learning, and to develop situations involving mathematical concepts. As Brousseau had been a primary teacher, the first elaboration of TDS (Theory of Didactical Situations) concerned primary level education and the tools it offered were focused on basic concepts in mathematics, such as numbers, operations (addition, subtraction, multiplication, division of whole numbers), random probability and geometry. Nevertheless, the intentions of the TDS theory are wider and it aims at a global organization and analysis of the teaching and learning context, in the field of mathematics. This model was designed to include at least three dimensions of the teaching-learning problematic (Brousseau 1997, p. 33):

- The first point is the pertinence of the description provided by the model, and the ability make evident the relevant phenomena in the field of research and experience;
- The second ambition is that this theory aims at the exhaustiveness in this description;
- The third point is the consistency of the analysis: Brousseau argues that teachers are not responsible for the coherence of the different tools they use in a classroom, but a theory must assume this coherence in its analysis of the field.

[1] Author's translation.

To study the knowledge to be taught, TDS introduced the concepts of *didactical transposition* and *fundamental situation* for a specific concept, supposed relevant for its emergence when adapted into sequences for a class. Then the difference between personal and institutional knowledge was introduced. From the beginning of the theoretical elaboration, the concept of "milieu" of a situation was used to characterize the mathematical and technical elements the students can actually use to solve the problem they face (alone or with the teacher's help). At the same time the concept of didactical contract was introduced by Brousseau, to specify the expectations about knowledge, explicit or not, of the teacher and the students towards each other. These concepts help the researcher both to understand deeply what occurs during the classroom, in terms of the teacher's and the student's work. For instance, the study of the contract makes possible the understanding of what could distort or reinforce the activities in which the pupils are engaged. These concepts also help to conceive new adapted situations, tied to fundamental situations and including some adidactical moments when the progress of the students' work may occur without the teacher's help (Brousseau ibid, cf. Article 6).

At the same time (the 70s up to the 80s), Regine Douady, working in the IREM of Paris Sud, elaborated her *tool-object dialectic model*, which is also focused on primary school. The concepts involved in her research are mainly tied to lengths, areas and decimal numbers for instance. Tool-object dialectic is a cyclic process organizing the role of the teacher and the pupils, in which mathematical concepts appear successively as tools for the solution of a problem for the students and as objects with a place in the construction of an organized knowledge, under the responsibility of the teacher. Douady proposed some tool-object situations. An interesting point of her work is also the idea that, to understand a mathematical concept, it is necessary to meet this concept in different settings, and to organize what she describes as an *Interplay Between Settings (IBS)*. As she says, "I.B.S are changes of settings (algebraic setting, numerical setting, geometrical setting) induced by the teacher in order to make the research of the pupils progress and their conceptions evolve." (Douady 1986, p. 5). It may be interpreted as a disequilibrium/re-equilibration introduced in the learning process.

All these theories have developed and been adapted to new contexts during the twenty-first century as we shall see below. But before that, a glance on the institutional frame seems important to better understand what occurs when it changes from 1993.

The first institutional structures of didactics of mathematics

Very important from the institutional point of view was the creation (starting from the 80s) of Didactics diplomas (for mathematics, and later for physics): DEA (equivalent of a master degree) and doctorate (PhD) with possible international collaboration. Moreover, research teams were created (Bordeaux, Paris, Strasbourg). Then, as informal teams became recognized laboratories, many researchers in didactics of mathematics were integrated in multidisciplinary laboratories. But there were no specific jobs for the didacticians in the university until 1990.

More precisely, starting from 1980, summer schools were organized every 2 years as also were, from 1977-78, national seminars (3 times a year). At the same time a research journal was created: RDM (from 1980), then another one: Annals of didactics and cognitive sciences. Some other reviews appear for educators and teachers: Grand N, Petit x.

International structures such as ICME, PME, CERME, EMF, etc., European Structures, English and Spanish journals have all come to enrich the diversity of this landscape.

Connections between research and school teachers

Generally speaking, school teachers are not directly involved as researchers in the didactical research, but there exist relations between teachers and researchers during the research involving some experiments in the classrooms or for in-service education. At the primary school level, the experimental school COREM (Centre of observation and research for the mathematics teaching) was created to enable research in classrooms and has been tightly associated with Guy Brousseau's team's research. Up until today, the COPIRELEM (Commission interIREM for the elementary schools) structure created in 1975 within the network of the IREM, enables educators to meet once a year and compare their experience and works.

At the secondary school level, along with the development of the IREM, training on new mathematics programmes allows even until today a collaborative work between some researchers, some educators and some teachers. However, the influence of the didactical research, which concerns all the education levels[2], is greater in the primary school than in secondary and in the university.

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[2] The first HDR in "pure" didactics (and not in mathematics) was held in 1981 and the subject was the acquisition of the concept of series' convergence. After a first thesis (PHD), giving access to the associate professor, the HDR is a "second" thesis giving access to the university professor ship (twenty years ago it was called "thèse d'état"). In a word-for-word translation HDR means "ability to conduct research".

2.2. Later stages in the development of the field of the Didactics of mathematics (80s-present)

New contexts

Beginning in the 1980s, developments of technology and software began to address the learning of mathematics, and some new issues emerged, tied to the integration of technology in class. Furthermore the expansion of the digital tools and computers led researchers to work on their integration in the mathematics classes. Their study as artefacts were introduced by some researchers (Artigue 2002; Trouche 2005). Taking into account the Internet, which changes the way teaching can be organized by introducing a greater part of collective work, led to the so-called Instrumental and Documentational approaches, that is, the study of the way teachers can have access to websites, documentation, etc. that can modify their way of preparing their lessons (Gueudet and Trouche 2009).

A favourable institutional context started in 1992: researchers in didactics were "welcomed" to participate in teacher education programmes for primary and secondary levels in the new institutional structure for educating young teachers (pre-service ones): the IUFM[3] in 1992 and then the ESPE since 2013. They could be recruited as associated professors and even full professors. For the secondary level it constituted a real extension of the training.

The development of international assessments, such as PISA or TIMSS, also led new research involving the possible didactical interpretations of these results, as relations between epistemological analysis and also new thinking about an effective use of the results for teachers (Chesné 2014; Grugeon-Allys 2016; Roditi and Salles 2015; Martinez and Roditi 2017).

Furthermore, as PISA highlighted for French students, in spite of several efforts, the inequalities between "poor and rich" children increased (not only in mathematics) and a lot of new research has been devoted to tackling this complex issue.

Development of the previous theories

The number of researchers in mathematics didactics is growing, and the research is increasingly organized by some theoretical frameworks. The main ones are still TDS but it evolves, depending on the contexts, and new ones develop too, leading to mixed approaches.

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[3] IUFM: Institut Universitaire de Formation des Maitres -University institute for teacher education – ESPE: Ecole Supérieure du professorat et de l'éducation -Institute for the teaching profession and education.

In TDS, new situations[4] have been designed for secondary or tertiary level, relating to functions, limits, irrational and complex numbers, integrals, linear algebra (in the work of Alson 1989; Bloch and Gibel 2016; Ghedamsi and Tanazefti 2015; Gonzalez-Martin et al. 2014; Haddad 2013; Lalaude 2016). Situations have also been analyzed in the context of students with special needs (see, for example, Bloch 2005; Voisin 2017; Favre 2015).

The notion of "*milieu*", to take another example, was widely developed (Margolinas 1995; Hersant and Perrin 2005), in particular to be adapted to the study of secondary school and university, and to contribute to the design of situations at this level (Bloch and Gibel 2016). For instance, the introduction of new levels to analyze such a situation allows researchers to better understand the emergence of what is expected in terms of proof in the setting up of an adidactical situation and to conceive suitable conditions for it. In Article 6, a precise analysis of this milieu is presented.

Some new didactical engineering (didactical design) has been developed and explored. New themes and methodologies appear and are 'more and more' developed: teaching practices (in regular classes), integration of ICT in education, second generation of didactical engineering, teachers' education, and assessment (recently).

The growing consciousness of the importance of mathematical symbolism leads to the introduction of semiotic components, for instance in a theory such as TDS (Bloch 2005). More researchers with different theoretical frames accord a new place to the study of the formalism in mathematics, their analysis refers to Duval's registers (1993), and even to Peirce's semiotic theory applied to didactic phenomena, extending the reflection on representations.

In the continuation of this work, Yves Chevallard developed another aspect of the study, an aspect which had not been taken into account in TDS: the institutional organization of the school system, related to mathematics teaching, and the way it works. Chevallard named his theory ATD: Anthropological Theory of Didactics, because it was inspired by anthropology, which describes and models the way human beings act in their society (Chevallard 1996).

ATD focused first on didactic transposition – the way mathematics knowledge is converted into different objects within the teaching process, and how teachers cope with this transformation. The mathematical reference analyses are based on the identification of the involved praxeology: this term addresses the classification of

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[4] The first research studies on these subjects were developed in 1980-90 but without consequences on university teaching.

human (mathematical) activity into types of tasks, techniques associated with the tasks, technologies (rationales of the techniques) in use and theories on which praxeology is implemented.

The ATD theory is deeply rooted in questions such as: which mathematics for which society, and how it is organized? The theory provides studies of the institutional context, curriculum, and processes of teaching/learning, according to the position of the studied human beings in the institution. It starts from an epistemological ground: mathematical knowledge analyzed in tasks, techniques, technologies (tied to justifications) and theories[5].

ATD has added new dimensions to the theory in the construction of Study and Research Paths (SRP) (see, Chevallard 2009): that is, problems for students joining a dimension of enquiry and, when possible, mathematical modelling of 'reality'[6]. The new context of teaching leads Chevallard to introduce the so-called "dialectic between media and milieu" to take into account both new resources, such as the Internet, and changes in students' scholar expectations. He claims for instance that it is essential to let students take advantage of the technological progress, as new means to question what is true or not in mathematics.

ATD has been used also by Sensevy to develop a theory about the didactic action of the teacher jointly with the students; let us notice that Sensevy, in cooperation with Assude (2009) and Mercier (Sensevy and Mercier 2007), also used TDS, and in particular the notion of the milieu, to analyze the joint work of teachers and students in a situation. Moreover, other 'local' theories have been developed, for instance a theory about different kinds of knowledge: CKC by Balacheff (Balacheff and Margolinas 2005).

2.3. The case of Activity Theory (AT) (and Double Approach - DA) in research in didactics of mathematics[7]

Emergence of Activity Theory in didactics of mathematics

A new focus on teachers' practices emerged, tied to the fact that a lot of researchers have more in mind the training of teachers, if only because of their professional activity, particularly for the secondary level. More precisely, according to their new professional missions, they have in mind the perspectives of teachers'

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[5] From the 2000's years (for instance Florensa, Bosch and Gascon 2015) it has been called a REM: Reference Epistemological Model.

[6] We can find for instance a SRP for the learning of 3D geometry (Petit x, 75), or other examples on: http://educmath.inrp.fr/Educmath/ressources/partenariat-inrp-07-OS/amperes/

[7] All this part 2.3 was partly published in the cahier du LDAR n°18, co-authored by Abboud, Robert, Rogalki and Vandebrouck (2017).

appropriation of some didactical analyses and results for their teaching. It does not mean that they imagine a precise training, but it means that many of them are more aware than previously of the question of transposition. For teachers, transposition implies what is at stake in didactics in teaching to achieve students' learning of the mathematics in focus.

Indeed , according to the well-known difficulty of the teachers to appropriate the results of research in didactics, and according to the French developments of the "professional didactics" (which go beyond the subject discipline), some researchers suggest that studying teachers' practices has to involve not only the aim of students' learning in the discipline but also the professional aims such as having peaceful classes, and so on. The didactic and ergonomic "double approach" of practices is related to this preoccupation, as it emphasizes the complexity of the teachers' practices with its consequences for (future) training. The main new goals concern the contributions to the study of these teaching practices, involving the study of what occurs in the classroom in terms of "possible" students' activities in relation to these teachers' practices (implementation studies). These goals were explicitly coming into the scope of Activity Theory, as already used in professional didactics, with an adaptation tied to the circumstances in practice. And this had another result in terms of AT: these researchers realized that the use of the AT was somehow implicit in the early 2000s for the analyses of the students' learning (Robert and Rogalski 2002, 2005; Robert 2012). Many tools regarding knowledge, teaching and learning, already developed in didactics of mathematics, may be used for AT's analysis. (It had been also applied to teachers (Rogalski 2003).

The inscription into this theory becomes explicit few years later (Robert and Rogalski, cited in Vandebrouck 2008, 2012; Rogalski 2012) for research into the activity of both teachers and students.

A use of the Activity Theory linked to the didactic and ergonomic double approach

These research studies' first focus is on students' learning in relation to the teaching that the teacher is deploying in the mathematics classroom (from primary school to the university).

To address this issue, researchers have chosen to study students' mathematical activities in the classroom: what the students do (or not), say (or not), write (or not). As what students think is not directly observable, researchers work on these activities' observable marks. This theoretical consideration is in line with the Activity Theory approach studying human subjects' activity, in practice, based on the distinction between task and activity. More precisely, it includes the fact that these activities are provoked (to a large part) by the teacher's activities in the working environment of the class.

Therefore, the research objects are the connections from students' activities to their learning (even if in fact it is tied to global hypotheses more than to accurate results) and from the teacher's activities to those of the students. The global aim is indeed relevant to give a diagnosis of what occurs in the class or to suggest some new ways of teaching that have to be explored. Hence, this approach is both experimental and theoretical, in a dialectical way, involving students' and teachers' observations and data collection, and also data analysis, including possibly new methodological developments.

But if the teacher's activity includes what is done before the class (conceiving the scenario and including some anticipation) and during the class (including some improvisation), these elements are not sufficient to understand the teacher's choices and their consequences on the student's activities. They also involve the professional experience, the knowledge and personal conceptions of the craft and of the mathematics to be taught. Researchers have to take into account also the way the teacher logs into institutional and social constraints (such as curricula, school's social environment and so on). This conception of the complexity of the teacher's practices characterizes the didactical and ergonomic double approach. This approach considers 5 components of the practices to interlink: two of them related to the choices of contents and implementation, two other related to the way the teacher takes into account social and institutional constraints and a last personal one, related to knowledge, experience and representations. Three levels to study the organization of the practices are added, which are related to each other. The global level involves the projects, the class designs, and so on, the local one involves what happens in the class (implementation, improvisation), and the third one, micro level, is devoted to the automatisms and routines. It particularly helps researchers to study the practices of beginning teachers, who have not yet global representations nor micro habits in the classroom.

But even if researchers are convinced of their importance, they do not systematically study parameters others than mathematics, tied for instance to affective factors, self-confidence, social and cultural origin. However a lot of research is devoted to the study of disadvantaged classes or schools at the elementary level. For instance, the research on disadvantaged classes went on, including the study of the teachers' practices and leading to descriptions of acute contradictions in those classes, between learning and quick achievement (for instance Peltier (2004), Butlen (2007). This research contributes to highlighting the frequent disequilibria in the classes between devolution (moments where the students work) and institutionalization (moments where the teachers address the knowledge to be learnt); the latter is often reduced, or even missed.

Taking these factors into account would for example involve for the students' activities some levels of organization, such as the global position posture according to the school, including the expectations and the relation to knowledge, the local attitude in class, including the participation to collective activities, and the micro level including some automatisms, for listening for instance.

New developments in the AT theoretical frame are conceiving tools to better target the distance between what students do and / or know and the teacher's actions and mediations according to an adaptation of the notion of a zone of proximal development (ZPD) for mathematics (proximity-in-action and discursive proximities). The aim is to cover the different ways of drawing on what the students already know or have done, more or less close to the general knowledge at stake. Some examples are given in Article 3. But also studying the moments of knowledge exposure through the development of analyses in terms of discursive proximities, in order to appreciate opportunities for possible or even missed proximities between what is general and stated by the teacher and what the students already know or do (cf. Article 3). The specific analyses of activities with technological tools allow access to what is new in terms of working on these instruments, both for the teacher and for the students and to provide the means to take more account of it (cf. Article 5). Unexpected difficulties of students have been brought up to date. Some of them are related for example to knowledge adaptations to be used by students when solving exercises. In some cases these adaptations are not detected by teachers and are left implicit. One could talk about teachers' "naturalization" when it is as if these adaptations are too familiar to teachers to be located. Often some students ask questions about these implicit adaptations, especially since the class is diverse. But if not, they may be overlooked and this likely blocks some students, even for a long time as it is often repeated.

Other developments are about the practices related to assessments, to collaborative research and the clarification of roles, to training and support of school teachers in very disadvantaged classes.

Let us notice that our "appropriation" of the ZPD notion must be specified, insofar as this notion is related to individuals whereas we use it in the context of a class.

3. Noting differences – results and outcomes

3.1. A key difference between the English and the French use of theory

The specificity of mathematics, mathematical knowledge, and mathematical thinking frames the French approach and constitutes the starting point of the French research. The building of mathematics teaching and learning processes and procedures on the basis of research on specific content is the common programme and development in the community. The English approach is perhaps to start from an understanding of teaching and learning in general, moving subsequently, or in parallel, to the specificity of mathematics. What is meant by ‘understanding’ is contested *within* the disciplines of psychology, sociology and philosophy, let alone *between* these disciplines. Hence, the English philosophical tradition of *pragmatism* has been pointed to as the framing of the proliferation of theories, and their relations to the practical traditions discussed earlier. We would argue that it is perhaps differences in how the work of the field is to be orientated between mathematics leading to teaching and learning, and teaching and learning leading to mathematics. More precisely, we now give a contrasting glance at the results, difficulties and perspectives in each tradition.

3.2. Results in English and French research

English research

In the French perspective, as seen above, there are three main headings. As we have set out in the English/UK section, there are many more theoretical orientations and indeed some of them are in opposition to each other. This makes it very difficult, if not impossible to identify ‘results’ that would be accepted across the community, and therefore it makes this an idiosyncratic account, dependent on the two particular authors of this part of our article. In this light we will summarise some ideas that we think most important to highlight, and in doing so we are perhaps looking to areas we consider have and are continuing to produce results, as well as those that are of major interest in the English community. We remind readers, however, that earlier theories are not replaced by new ones but continue their ‘internal’ development. We list five, below.

Strong centralized regulation and policy studies

Where there have been policy studies in mathematics education they have revealed the strong hold on what is taught, how it should be taught, and how learning is measured by the Education Department of Government of both left and right. The reports from the powerful and influential framework of inspection of schools are taken in place of research to inform policy. Performance in mathematics in PISA and TIMSS reports that have shown an apparent slip in UK achievement over the years are another element in what informs policy. The critiques produced by the research community (see e.g. Lerman and Adler 2016), inevitably, do not impact on Government. Nevertheless we consider that such studies by mathematics education researchers, revealing the negative effects of such control, and positive ones where they appear, are needed.

Studies into classroom practice uses of technology and mathematical understanding

The relationships between classroom practices and the theories used to analyze and explain didactical and pedagogical approaches to creation of mathematical understanding are still central to English research and relevant to classroom practice in the UK. Practical traditions are alive and well, pursued by teachers and teacher educators, in relation to political forces and school organizations. Research results inform such practice. A central interest is in developing practices which achieve students' mathematical understanding. As indicated above, a range of theoretical perspectives are used by researchers in these areas. The use of technology is now firmly embedded in the curriculum, but research and associated theory into this use varies with focus. Much research is very small scale with teacher educators and teachers exploring situations at a local level and using theory as it seems to support their own research questions and design. Both constructivist and sociocultural theories, as well as enactivism and instrumentalisation are used. Large scale projects are few, due to limited sources of funding and the requirements of funding bodies.

Informing equity studies

The replication of social class differences in terms of achievement in school mathematics remains an intractable problem. The main factor associated with success and failure in mathematics remains family socio-economic status. In the research field there are insights that offer ways forward, such as gender studies, critiques of setting[8], teacher expectations of who can achieve in mathematics, and working with challenging mathematical problems, but whilst many teachers have adopted and use the materials that have developed from research, these in general have not been taken up in curricula or learning goals as prescribed by Government. We should note here, although the evidence comes from outside of the education research community, that achievement overall, including in mathematics, has improved for all children, though to a lesser extent for children from low socio-economic groups. This has been achieved in some areas of the UK, particularly London. The causes are likely to do with levels of investment, embedding of higher expectations of all children, and other factors outside of the research field of mathematics education.

Meaning and relevance in mathematics

In the mathematics education research community internationally there are growing numbers of studies of the role of everyday reality of students to be brought into the classroom to make mathematics relevant and meaningful. Theoretical perspectives developed include ethnomathematics, critical mathematics and 'funds of knowledge' (see Civil 2016). In the English research, critiques of these approaches have come, in the main, from theories in sociology of education.

[8] The grouping of students according to some concept of 'ability'; thus producing a hierarchy of 'mathematics sets' in a school year group (see Boaler and Wiliam 2001).

Drawing on Bernstein or Bourdieu (e.g. Cooper and Dunne 2000), researchers have identified how turning to the every day to provide meaning that may motivate students better than the decontextualised mathematics that predominates in textbooks and curricula may indeed further disadvantage students who do not succeed in school. These studies have, we believe, pointed to important issues in learning mathematics in general and for students from disadvantaged backgrounds in particular. This is not seen to devalue ethnomathematics, etc., but to indicate the tension between the relevance and applications of mathematics and gaining a grasp of the esoteric symbolization of mathematical knowledge and the need for both.

University mathematics

Finally, to point to the growing body of research on the teaching and learning of mathematics at University level. For rather too long the research community has treated this field as unproblematic. The recent decade or so of the growth of this field has shown that this view is quite wrong. Cognitive studies explore the mathematical learning of large numbers of students, whereas socioculturally rooted studies seek insights into pedagogical practices and their impact on learning outcomes. A main difference with these studies and those conducted at classroom level relates to the number of students in a particular cohort (often between 100 and 300) and an economic need to teach them all together. Thus research results into practices at school level (where class sizes are around 30) are often not applicable at this higher level. The research findings and their implications are informing University mathematics staff, though there are still many who place the whole reason for student failure on the students themselves.

French research

We now summarize some salient features of the main research in the French tradition.

Theory of Didactical Situations

The framework TDS is particularly concerned with the design of learning situations of which the implementation has to be studied. Some evolution occurred in the research so that now the ordinary classes and the resources production are also studied. But the main aim remains to study the cognitive potential of a given situation, that is the study of what the students may learn according to the choices of mathematical content. This leads to the identification of what could be due to the *milieu* (present in the situation independently of the teacher) and to the didactical contract, but it also leads to work on the didactical variables that enable the teacher to play on possible student actions. This induces a conception of the actors as generic subjects, having a specific function (student, teacher) rather than as singular, active subjects as it is for AT and double approach.

However, research studies remain mostly at a local level of analysis, even if the curricula are obviously taken into account; depending on the adopted framework, students and teachers are considered as more or less “generic”.

Anthropological Theory of Didactics

The ATD, concerns more a global vision of the mathematics education system including teachers, emerging from existing constraints and norms. Moreover, the phenomena identified are related to different levels of determination, ranging from class to society. This leads to a conception of the actors as subject to a given institution, and, again, not as singular actors.

ATD allows us to:

- study the modifications of the institutional context, e.g. how it works in a professional environment such as in Engineering schools, and also which mathematics are taught and why;
- take into account the conditions and constraints of teaching, and analyze the teachers’ role, for instance how they introduce and validate tasks, techniques, technologies and theories $\rightarrow$ in algebra, geometry, calculus; what is the relation between these *praxeology* (cf. above) and the school level; how these conditions can appear in didactical studies and how they can be taken into account in teachers’ training;
- build “inquiry-based teaching” as SRP (cf. above).

Activity Theory

By giving *a place to students and teachers in their singularity*, as “human beings”, the AT framework is specifically adapted to study what effectively happens in class, whether practices are ordinary ones or not. Local analyses are more developed than the global ones.

In terms of results, researchers can stress obtaining important results on the teacher's practices and on their stability[9] (shown by several of our research studies), ensuring therefore the validity of the extension of our local outcomes. Taking into account contents and constraints, some “robust” scenarios have been proposed and tested. “Robust” means that whatever the implementation in the classroom is, if not extraordinary, the expected activities are possible for students.

This research also enables researchers to propose a critical view of institutional instructions. Tensions may exist for instance between diverse expected rigour

[9] For experienced in-service teachers, the component tied to the implementation choices seems particularly stable.

requirements needed for the different contents that have to be taught during the year. The didactical contract may be raised to describe those requirements, but AT research addresses the issues on teachers' practices in terms of choices and students' difficulties. Another example is relevant: in order to give sense to mathematics, the institutional injunction makes the students work on complex tasks. But to solve these complex tasks, students use diverse procedures. Then this diversity makes it difficult for teachers to highlight the knowledge aimed for between the various paths that have been used.

Finally, it is important to be aware of the fact that French theories, whatever they may be, are by no means a sort of enclosure but rather guarantors. It is important to use them to guarantee a certain coherence in the division of the observed reality, but also to identify what could be unexpected, and even to know how to transform what first appears as a "disturbing noise" into a new development. Likewise, if data gathering must be adapted to the theoretical frameworks, this, fortunately, may still produce unexpected phenomena; these are opportunities that the research has to grasp!

Conclusion

It is quite impossible to compare general uses of theories since, despite the many differences articulated above, the deeper issues and outcomes of activity in the two domains, such as mathematical understandings, impact and scale of research findings, are not so different. Articles 3 and 5 in this volume, are devoted to research on some key issues and allow us to understand more deeply how the theories are used or developed in each case. We address successively examples of different uses of AT as a lens to study what occurs in a classroom, what occurs with uses of Digital Technology in mathematics teaching, and of how practice and theory are related in the use of video in teacher training contexts. Article 6 provides an insight on the use of TDS in two contexts of training and lets us understand their importance, relative to each outcome. In the concluding article 7, we take up again the ways in which our research approaches are different or comparable and ends with a glance towards the biggest issues that face us all.

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