CHRISTINE MANGIANTE-ORSOLA, MARIE-JEANNE PERRIN-GLORIAN, HEIDI STRØMSKAG

THEORY OF DIDACTICAL SITUATIONS AS A TOOL TO UNDERSTAND AND DEVELOP MATHEMATICS TEACHING PRACTICES

Abstract. This article aims to discuss how the theory of didactical situations in mathematics (TDS[1]) can be used to answer research questions concerning regular teaching practices, production of resources for regular teaching, and teacher development. In the first part we focus on TDS and the way it may be a tool for the researcher to understand teaching practices and the way it may contribute to develop teaching practices, helping teachers identify questions useful for their practice. In the second part, we present analyses using TDS in two contexts in which researchers worked with teachers, making explicit or not the concepts they used. The third part approaches, from these two contexts, the way TDS may help the collaboration between researchers and teachers (or teacher educators), in research on teacher development, in particular in the case of producing resources helping teachers to prepare their class. The comparison of the two contexts informs on the specific contribution of TDS in understanding and developing mathematics teaching practices.

Keywords. Theory of didactical situations, mathematics teachers' practices, teachers' development, resources for mathematics teachers, multiplication, geometry.

Résumé. La théorie des situations didactiques comme outil pour comprendre et développer les pratiques professionnelles des enseignants en mathématiques. Le but de cet article est de discuter l'utilisation de la théorie des situations didactiques en mathématiques (TSD[2]) pour répondre à des questions de recherche concernant les pratiques ordinaires d'enseignement, la production de ressources pour l'enseignement ordinaire et le développement professionnel des enseignants. Nous centrons la première partie sur la manière dont la TSD peut être utilisée par le chercheur comme outil pour comprendre les pratiques des professeurs et comment elle peut contribuer au développement de ces pratiques en aidant les professeurs à identifier des questions utiles pour leur pratique. Dans la deuxième partie, nous présentons des analyses appuyées sur la TSD dans deux contextes dans lesquels les chercheurs ont travaillé avec des enseignants en utilisant la TSD, en explicitant ou non les concepts utilisés. La troisème partie aborde dans ces deux contextes la manière dont la TSD peut aider la collaboration entre chercheurs et enseignants (ou formateurs) dans les recherches sur le développement des pratiques enseignantes, notamment dans le cas de la production de ressources pour aider les enseignants à préparer la classe. La comparaison des deux contextes permet d'éclairer

[1] In the more recent texts, Brousseau specifies "in mathematics", speaking of the theory of didactical situations. Nevertheless, for short, we use TDS, which is more usual.

[2] Dans les textes récents, Brousseau spécifie « en mathématiques » quand il parle de la théorie des situations didactiques. Nous utilisons néanmoins l'abréviation courante TSD.

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l'apport de la TSD dans la compréhension et le développement des pratiques des enseignants en mathématiques.

Mots-clés. Théorie des situations didactiques, pratiques des enseignants de mathématiques, développement professionnel des enseignants, ressources pour les enseignants de mathématiques, multiplication, géométrie.

Introduction

Unlike the other articles of the special issue, this article refers mainly to one theoretical frame. Indeed, our purpose is to discuss how the theory of didactical situations in mathematics (TDS for short) can be used as a tool to understand and develop teachers' mathematics practices, that is to say all that teachers have to do in order to carry out the mathematics teaching in class in all its complexity: planning, designing, implementing, analyzing and validating teaching units.

TDS emerged in strong interaction with a methodology of didactical engineering and developed concepts and models helping conceptualize the evolution of mathematical knowledge (from informal mathematical knowledge to formal, mathematical knowledge), and identify the teacher's roles in different phases of this evolution. Later, some researchers (e.g. Hersant and Perrin-Glorian, 2005; Margolinas, Coulange and Bessot, 2005) used it to study regular teaching with a nearly naturalistic observation. We discuss here its relevance in the development of teaching practices and in research on this development in two different contexts, the first (case 1) on multiplication with Grade 3 students in Norway, and the second (case 2) on geometry with Grades 3 to 5 students in France.

The contexts differ not only by the mathematical content at stake but also by the purpose of the research in which each of them takes place, by the way researchers and teachers collaborate and by differences concerning teacher education and teacher recruitment in the two countries.

In case 1, the data come from a four-year intervention project in Norway, LaUDiM (Language Use and Development in the Mathematics Classroom) (Rønning and Strømskag, 2017) in a context of pre-service teacher education. The teacher training goal is to help teachers design, implement and analyze a teaching situation where there is an intention of teaching primary students some particular mathematical knowledge (here, multiplication) that could be perceived as meaningful for the students. The research goal was threefold: to design a teaching situation for third graders' first encounter of multiplication based on *a priori* (epistemological and didactical) analyses; to observe the situation implemented in class; and to validate the situation in terms of comparison of *a priori* and *a posteriori* analyses. One teacher, with certain awareness of some concepts of TDS is involved in direct collaboration with researchers who are at the same time teacher educators[3] for pre-service teachers, with whom they will use the results of the analyses.

In case 2, the data come from a research project in France gathering two researchers and five teacher advisors[4] from one educational district for primary schools (about 200 classes). The teacher training goal is to help teachers think about geometry teaching in Grades 3 to 5 (8-11 years old) and to produce reflection and resources to help practicing teachers in this teaching. The research goal was threefold: to elaborate an organization of the teaching of geometry coherent from 6 to 15 years; to work out with teachers advisors a resource for teachers coherent with our assumptions about geometry teaching; and to investigate the way teachers of primary school, not specialists of mathematics, may develop their geometry teaching using this resource. Twelve regular teachers are associated to the project: they implement in their class the situations first designed by the researchers and the teacher advisors. Neither the teachers nor the teacher advisors are aware of concepts of TDS, except perhaps the one of didactical variable.

In Section 1 we focus on the way TDS may be a tool for the researcher to understand teaching practices and to help teacher development. Section 2 presents the two contexts and the analyses with TDS. Section 3 approaches, from these two contexts, the way TDS may help the collaboration between researchers and teachers (or teacher educators), in research on teacher development. Then, we come back to the comparison of the two cases, in relation with the use of TDS to clarify how this theoretical frame can enlighten teaching practices and we draw out some questions for more investigation and articulation of TDS with other theoretical frames related to Vygotsky's work or Activity Theory.

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[3] In Norway, most mathematics teacher educators are researchers. The ones participating in the reported research are all researchers. In schools where pre-service teachers have their field practice, there are mentors supervising pre-service teachers' practice in their own class. The mentors are teachers who contribute to teacher education, but they are not researchers, and they have no teaching duties at campus.

[4] In France, most mathematics teacher educators are not researchers. Teacher advisors are teachers of primary school partly or totally without a class. They contribute to teacher education (as mentors or for in-service teacher training). Three among those who participate in the research reported here have a class during two thirds of their work time and contribute to pre-service teacher education; the two others are advisors without a class of their own and contribute mainly to in-service teacher education.

1. How TDS may help the researcher to understand teaching practices?

The theory of didactical situations in mathematics provides scientific concepts that allow one, researcher or teacher, to understand or predict certain didactical phenomena in any situation in which there is an intention of teaching someone a particular piece of mathematical knowledge, whether they succeed in it or not. In regular teaching, TDS allows the analysis of an actual opportunity for a student to learn and gives means to provide such an opportunity. It was elaborated by Brousseau mainly during the 1970s and 1980s, with a methodology of didactical engineering (Brousseau, 2006). During the 1990s, Brousseau stressed the importance of the notion of *milieu* in the theory (Brousseau, 1997b, 2000) and he developed the notion of *didactical contract* (Brousseau, 1997a, 1997b) and insisted, on many occasions, on the fact that TDS is able to represent any situation in which there is an intention of teaching someone some specific mathematical knowledge. More recently (Brousseau, 2000; Perrin-Glorian, 2008) students' learning is seen in TDS as a combination of two processes (see Figure 1).

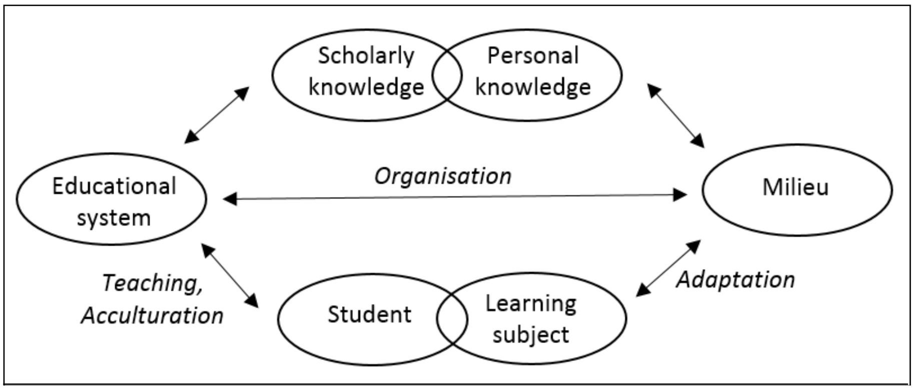

Figure 1. The didactical situation in TDS (translated from Brousseau 2000)

On the one hand, independent adaptation to a milieu (conceptualised through an *adidactical* situation) and on the other hand, acculturation into an educational system (through didactical situations and contract). In this model, the devolution ensures the conditions for adaptation, and the institutionalization ensures the conditions for acculturation. At the same time, TDS became to be used to study regular teaching with a methodology of class observations, with as few interventions of the teacher as possible, in the preparation of the class (Hersant and Perrin-Glorian, 2005; Margolinas, Coulange and Bessot, 2005). TDS was then a tool for the researcher to understand teaching practices by posing questions for observation and analysis of these practices. Answering these questions makes it possible to understand how knowledge can progress in class and who contributes to this progress.

1.1. A brief presentation of TDS

The methodological principle of TDS involves implementing target knowledge in a situation that preserves meaning; that is, the target knowledge appears in some sense as an optimal solution to the given problem. If the teacher succeeds in making a devolution of this problem, that is the problem is taken over by the students as their own, it provides a *purpose* for the students to engage in the situation, and the target knowledge appears as meaningful and useful (what it can be used *for*) because it solves the problem in the situation. The following diagram (Figure 2) recalls the main issues of TDS to represent a didactical situation, focusing on the teacher with the perspective of understanding how the students learn and how the teacher helps them learn some mathematical content with the help of this situation.

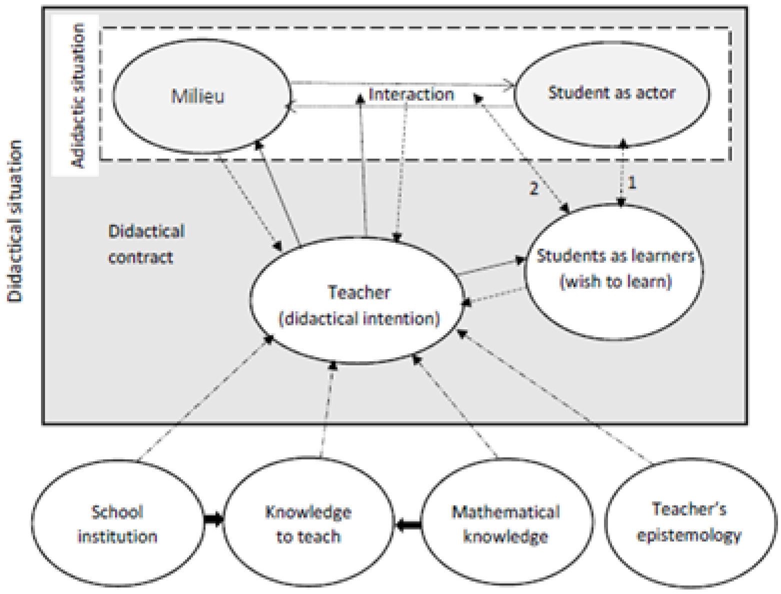

Figure 2: Interactions in a didactical situation (in the sense of TDS)

The didactical situation is represented by the grey rectangle. In this situation, there are two kinds of actors: the teacher with an intention to teach some mathematical object and the students; they are linked by the didactical contract. The white rectangle inside the grey one (with a dotted edge) represents the adidactic situation we can identify inside the didactical situation, as a way to learn a new piece of mathematical knowledge: a generic student, representing any student, acts on a milieu[5] that is able to give feedback on those actions. The adidactic situation may be considered as a game[6] defined by this milieu, rules to interact with it and an aim to reach: how to win. It is constructed or chosen by the teacher such that the knowledge to win will be the knowledge to be learnt and the prior knowledge of students may help them to play the game and interpret the feedback of the *milieu*.

These conditions can be expressed by three constraints on the *milieu* (Salin, 2002): (1) to provoke contradictions, difficulties for the students so that they have to adapt their knowledge; (2) to allow them to work autonomously; (3) to help them to learn some specific mathematical content (by learning to win the game). Thus, to learn, the student has to play the game (acting him/herself or in interaction with others), following the rules (and his/her own idea) and reflect on this action taking into account the feedback of the milieu, whether s/he won or lost.

Black arrows: The teacher interacts with the milieu (to construct it before the class or to modify it during the class), eventually with the relation between the actor and the milieu to change the game (with an aim of devolution[7] for instance) or on the students' knowledge (institutionalization for instance).

Dotted arrows: The teacher takes information on the relationship between the student and the (adidactic) milieu, on the students' knowledge (in act or expressed). S/he will be able to use this information to modify the milieu or to give some help to some students. The students as learners consider the action on the milieu (arrows 1 and 2) and reflect on it as a way to produce new knowledge. These actions may be indirect or implicit (not easy to observe).

Arrows with short lines and dots (at the bottom of Figure 2) represent constraints and objectives of the teacher, coming from the school institution or her/himself. Knowledge to teach is interpreted by the teacher from the curriculum and her/his

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[5] The *milieu* represents the elements of the material and intellectual reality with which the students interact when solving a task. These elements may comprise: material or symbolic tools provided (artefacts, informative texts, data, etc.); students' prior knowledge; other students; and arrangement of the classroom and rules for operating in the situation. For a very short presentation of the notion of milieu, see (Perrin-Glorian, 2008). Examples will be found in the second part of this article.

[6] *Game* is a metaphor – it has to be understood in a theoretical sense, as a model of the problem to be solved with related conditions.

[7] *Devolution* and *institutionalization* are two components of the game that the teacher has to play so that the student learns from the situation. In devolution, the teacher acts so that the student plays the game to win and not to please him/her. In institutionalization, the teacher's aim is to help the students recognise the knowledge gained in the game and to transform it into knowledge usable to solve other problems.

own mathematical knowledge. We do not represent constraints on students, though they exist, coming for instance from their parents or from other students.

1.2. TDS to analyze regular mathematics teaching

The description above, of a didactical situation in TDS, gives a researcher means to observe and analyze a regular teaching class session constructed by a teacher without the help of the researcher because it gives questions to pose, in order to define these elements from the class session observed: the adidactical part of a didactical situation in the sense of TDS (problem and milieu), as well as the didactical contract; to carry out the a priori analysis (i.e. analysis of what was possible) of this situation and to compare it with the a posteriori analysis (i.e. analysis of what actually happened). Of course, to answer these questions we need previous analyses involving the knowledge to teach (e.g. epistemological analysis and analysis of the curriculum) and the previous knowledge of the students. For example and details, see (Hersant and Perrin-Glorian, 2005)).

We can summarize some of these questions as follows :

1. What is the didactical intention of the teacher (the mathematics knowledge s/he wants the students to learn)?
2. Can we identify the objective milieu provided for the students? By objective milieu we mean here all the data independent of the teacher’s interventions and from the students’ knowledge afforded for the action or reflection of the students.
3. Is there something problematic for the students in this milieu? How may they solve this problem? What knowledge is at stake for the students? What use of knowledge is necessary to interact with the milieu and solve the problem? (Is it needed in order to: Progress in finding a solution to the problem? To formulate the solution in such a way that somebody else be able to solve the problem? To prove that this solution is a good way to solve the problem?)
4. What is the status of this knowledge for the students (quite new knowledge, knowledge in the course of learning, knowledge supposed known)? In this question we include the relations between knowledge at stake (new or old) and the didactical contract (what is expected from the teacher, from the students) in the domain.
5. What are the choices in the milieu that the teacher can change so that the knowledge at stake for the students changes (i.e. didactical variables)?

These questions may be posed with different scales: at the meso-scale of a sequence of classroom sessions or of one lesson; at a macro-scale of the insertion of this sequence (lesson) in the teaching of a mathematical domain; at the microscale of interactions between the teacher and the students. At the micro-level, the milieu evolves in the course of the lesson after some actions of the students or of the teacher. Thus we use the notion of 'situation' at different scales too. Usually, we begin with the meso-level of the class session including it in a more macro-level of analysis for the knowledge at stake and we consider the micro-level only on some parts where we find something happening in the perspective of the progression of knowledge for the students (progress or difficulty).

Answering these questions helps define a situation in the sense of TDS and provides an understanding of how the knowledge can progress in class. Moreover, to understand who contributes to this progress, we add some other questions concerning the relationships between what the students do and what the teacher does.

1. *Devolution*: what does the teacher do so that the problem becomes each student's problem all along the session?
2. *Regulation*: what does the teacher do so that the students work really on the content at stake? How does s/he help them?
3. *Institutionalization*: what does the teacher do so that the knowledge used to solve the problem becomes a piece of knowledge to know and to use in other situations?

Clearly, answering these questions depends strongly on the knowledge to be learnt. We are particularly attentive to the different meanings likely to be attributed to the word "knowledge" even if we consider a specific item knowledge in mathematics. From the knowledge, as s/he knows it, and from its definition in the curriculum (knowledge to be taught), the teacher has to choose problems where this knowledge is useful (as knowledge to act in the problem) and to define what s/he wants the students to be able to do with this specific knowledge (knowledge to learn for the students), and then what they actually learnt and are able to do with it (knowledge actually learnt).

To specify some of these questions and answer them, it may be useful to connect TDS with other theoretical frames, on the one hand to analyze the knowledge at stake, on the other hand to analyze the teacher's action as we shall see with the two examples in the next parts of this article.

1.3. How TDS may help teacher development?

From the point of view of the teacher development, the concepts of TDS may help identify questions useful for the teacher in three moments: in the preparation of the class; during the lesson; in analysing what happened.

The concepts of TDS, mainly those of milieu, didactical variable, action, formulation, validation, devolution, regulation, didactical contract, and institutionalization are quite important for the action of the teacher but it is not really necessary that s/he knows them in a theoretical way (as concepts of a theory) to be able to use them in practice. S/he can access these concepts to analyze and improve her/his practice for instance by a collaboration with a researcher in observations and analyses of situations in her/his classroom or in other classrooms. The teacher needs to relate these concepts to her/his concrete practice, what s/he usually does to prepare or analyze the lesson.

2. Using TDS to help teacher development in two different contexts

Our intention in this section is to present the use of the underlying concepts of TDS through two case studies in primary school: teaching of multiplication in Norway; and, teaching of geometry in France. In the two contexts research questions concern teacher education. TDS intervenes at two levels: 1) How can it help to enlight teachers' practice and be useful in teacher training? 2) How does it contribute to the researchers' methodology and analyses? In this section, we give first a description of the class sessions in the two contexts and then some examples of the use of TDS to analyze the teachers' practices. Questions linked to this use, according to research questions and the different ways teachers and researchers interact in the two cases, will be discussed in Section 3.

2.1. Presentation of the data in the two contexts

The case of multiplication

This section is a description of a teaching sequence on multiplication in a Norwegian Grade 3 classroom (18 students, 8 years old). Records were gathered of: pre-analysis and planning (in a team of a class teacher and five university researchers, one of whom is one of the authors of this paper); two classroom sessions; and a reflective meeting (in the team) after the first session. The researchers who took part in the planning were teacher educators of mathematics. In the project team there were also two pedagogues (general educators) who were researchers, and another teacher. The observations were video-recorded, and the reflective meeting was audio-recorded. TDS was used implicitly to design the sequence on multiplication.

Pre-analysis and planning

The described teaching sequence was the students' first encounter with multiplicative structures. In preparation for the pre-analysis, all in the team had read an article by Greer (1992), where he proposes that the most important types of situations where multiplication of integers is involved, are: equivalent groups (including rate); rectangular arrays; multiplicative comparison; and Cartesian products. The team agreed that the focus should be on situations with equivalent groups (i.e. of the same size) and rectangular arrays. Researchers suggested that the target knowledge was understanding situations with equivalent groups in terms of multiplication, and being able to write the result as a product, where for example 5·3 would be explained as "five threes", or "five times three", or "five groups with three (objects) in each group". The teacher said that a goal for her was that the students should *write an arithmetic problem*[8] that fitted with the task. "For instance, if Pauline[9] has five bags with three apples in each bag, how many apples does she have all together?" (quoting the teacher). Here the teacher would like students to write 3+3+3+3+3=15 (not 5+5+5=15) to say 5 sets of 3 apples gives 15 apples, which she would subsequently institutionalize as 5·3=15.

It is relevant to notice that in Norwegian schools, multiplication is usually introduced through situations with equivalent groups, where conventionally, 3·5 means 5+5+5, while 5·3 means 3+3+3+3+3 (i.e. a model of repeated addition). It was pointed out that multiplication as an operation is commutative, whereas situations to be modelled by the operation can be either commutative or non-commutative. Both types of situations were exemplified.

Based on the pre-analysis and planning, the teacher made a set of three tasks in the form of word problems. Classroom work on Tasks 1 and 2 (presented below) will be described and analyzed in this paper.[10]

Task 1. Class 3c plans to arrange a class party in the Café. The day before the party, they will bake muffins for the party at school. Pauline has to go the grocery store to buy eggs for the muffins. The recipe says there should be four eggs in one portion. The students have decided that they will bake twelve portions of muffins. How many eggs should Pauline buy?

Task 2. The muffins are placed on baking trays to be baked in the oven. On a tray there is space for five rows of muffins, and there is space for seven muffins in each row. How many muffins can be placed on one tray?

The teacher's didactical intention was: (1) equivalent groups put together should be interpreted in terms of multiplication as repeated addition; and (2) the problem in the task should be written as a product, where the first factor in the product signifies the number of groups (multiplier) and the second factor signifies the size of the groups (multiplicand). The situations in Tasks 1 and 2 are multiplicative

[8] 'Arithmetic problem' is translated from Norwegian 'regnestykke'; it means to write what calculations are needed to solve the problem.
[9] All names used in the paper are pseudonyms. Pauline is the teacher.
[10] Task 3 (on multiplicative comparison) was not reviewed in the analyzed sessions.

structures that consist of a simple direct proportion between two measure spaces, a structure referred to by Vergnaud (1983) as isomorphism of measures. The situations in the two tasks are however different in nature: The first situation (Task 1) is non-commutative, where one factor measures a number of iterations and the other measures a magnitude; this type of situation is understood as equivalent groups. The second situation (Task 2) is commutative, where the two ways of making iterations for counting are equivalently natural; this type of situation is understood as a rectangular array.

Implementation of the tasks

The teacher explained to the students that they would work in pairs on three tasks about an imagined class party at school. She said that she wanted them to draw on sheets how they would solve each task, and that, later, two pairs would be put together to explain how they had solved the tasks. After the students had made drawings and found the answers by counting, the teacher asked them to write “arithmetic problems” that showed the calculations. Later, she initiated a transition to the phase where two pairs explained their solution to either Task 2 or Task 3 (Task 1 was not part of this sharing). At the end of Session 1, the teacher gathered the students at the interactive white board to enable sharing of how they had solved Task 1. She invited them to the board (one at a time) to write and explain their methods. Below, two solutions to Task 1 are shown (Figures 3 and 4).

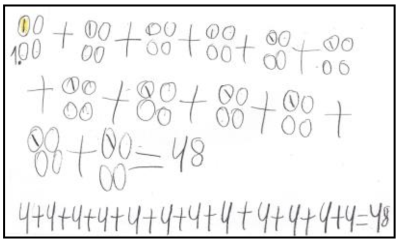


Figure 2. Lucas and Nadia’s solution

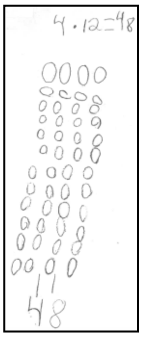


Figure 3. Filipa and George’s solution

The teacher concluded Session 1 by referring to the product 12·4 (which was the solution by only one pair, Filipa and George)[11], and said that they would look closer at 12·4 in the next session. That is, her goal for Session 2 was introducing product notation.

[11] At first George had written 4·12 (see Figure 4) , but changed it into 12·4 after some input from one of the researchers. George was the one who had written 12·4 on the board.

Right after Session 1, the team had a short meeting to reflect and possibly make adjustments for Session 2. The teacher referred to the situation with portions and eggs, and said that it was challenging to sum up at the end, the matter with the order of the factors in a product, and what the factors mean. She commented that it was not possible to swap the factors in Task 1, without losing the meaning of the situation. The team discussed how the situation might be reinterpreted[12]. The teacher described how Task 2 was different from Task 1: For muffins on a baking tray, rows and columns can be interchanged. She decided to use Task 1 to establish the *convention* of the order of the factors, and Task 2 to establish *commutativity*.

Two days after Session 1, the students were gathered at the board, where the teacher reminded them about Task 1, using the image in Figure 5.

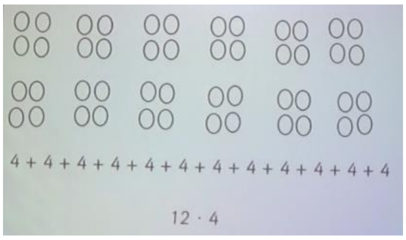


Figure 4. Multiplication as a model of an equivalent-groups situation (Task 1)

It addressed primarily those who had made distinct groupings in fours; there was no mention of the sequence 4, 8, 12,…, 48 (given by two pairs). The teacher's goal was that the students should learn to write 4+4+4+4+4+4+4+4+4+4+4+4 as 12·4. She asked why the repeated addition fitted with the situation. Nora answered that it was because it should be four eggs 12 times. In the discussion that followed, there was some back and forth about the interpretation of 12·4: "twelve times four", "twelve four times", "four twelve times". The teacher wrote on the board 12+12+12+12, and 4·12 under it. Then she asked about the distinction between 12·4 and 4·12. Tanya answered "If four is written first, then you shall take 12 four times, and if 12 is written first, then you shall take um… four 12 times." Some reinforcement of Tanya's interpretation followed.

The discussion continued as the teacher asked why writing 12·4 is "smarter" than writing 4+4+4+4+4+4+4+4+4+4+4+4. Responses suggested it is faster than writing all the fours. However, one student pointed out "We wrote it fast too, with plus." The teacher responded by supposing that they were making a thousand portions of muffins—what would this be? Students replied "a thousand fours", and

[12] 4·12 could be interpreted as 4 groups of 12, where the first group consists of the first egg from the 12 portions, the second group consists of the second egg from the 12 portions, and likewise for the third and fourth groups. This was not meant to be presented to the students.

that it is “a thousand times four”. But Lucas argued “Now you take a thousand four times”. He explained that he just “turned it” and took 1000 plus 1000 plus 1000 plus 1000, and got 4000. The teacher said that this was right, and that there are some smart ways of calculating this, without explaining this further at that time.

Afterwards the teacher turned to a review of Task 2, using Figure 6 as an illustration.

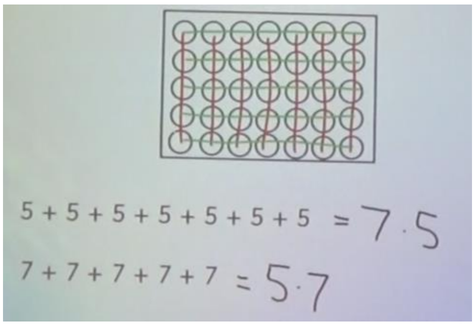


Figure 5. Review of Task 2

The horizontal and vertical lines and the products were inserted during discussion. She showed (what several pairs had pointed out in their solution) that they would get the same number of muffins, whether they counted 5+5+5+5+5+5+5=7·5 (vertical lines in Figure 6), or 7+7+7+7+7=5·7 (horizontal lines).

In the above, two different situations were aiming at multiplication as a model: first, an equivalent-groups situation (portions and eggs), then a rectangular-array situation (rows/columns and muffins). There was no discussion of any connection between the situations.

In Section 2.2 we present a TDS analysis of the sequence (done by the author involved in the project), the aim of which is to identify issues for development of teaching practices.

The case of geometry

A collaboration between researchers (two of the authors of this paper[13]) and teacher advisors was carried out for several years in order to reflect on geometry teaching in grades 3 to 5 (8-11 years old) in France and produce reflections and resources to help teachers in this teaching (Mangiante-Orsola and Perrin-Glorian, 2017). With this aim, we designed situations that were implemented first in the classes of the teacher advisors who had one, discussed, and then proposed to a group of about twelve teachers who implemented them in their classes. The sequences were

[13] They are (or were) at the same time teacher educators.

observed by the teacher advisors eventually accompanied by one researcher; some of them were video-recorded. The data were discussed first in the small group of researchers and advisors, then in the large group with all the teachers. Our approach to geometry rests on the the work of a research team in the North of France from 2000 to 2010 (Duval, 2005; Perrin-Glorian and Godin, 2014, 2017). A main construct is the vision of figures: the natural vision of figures is a vision of juxtaposed surfaces; in mathematics, geometrical figures are defined by relations linking lines and points so that you have to focus your gaze on these components of the figure instead of viewing the figure as a combination of surfaces, as comes naturally to the eyes.

The main idea to build problems for the students is to make them reproduce figures under certain conditions, what we called *"restoring a figure"*. To restore a figure, students have a model figure (always available) and a *beginning* of the figure to reproduce (small part already reproduced, the same size as the model or a different size). They may use tools (usual geometrical tools except tools for measurement[14], but also non conventional tools, such as templates) to take information from the model (for this, they are also allowed to trace on the model figure) or to draw the new figure. When they have achieved their reproduction, they may check it with the figure to be drawn, on tracing paper. Roughly speaking, the *milieu* is constituted by the model figure, the beginning of the reproduction, and the tools available. The game consists in reproducing the model with the tools. You win the game if the figure on the tracing paper exactly fits with your reproduction. The choices of the model, and the beginning and the tools are *didactical variables* because the knowledge necessary to achieve the figure strongly depends on them.

In this paper, we focus on one crucial situation of the sequence. The objective was to help teachers think in a different way about geometry teaching while proposing to them a situation for the class to exercise the way of looking at a figure and to work on the notions of alignment, line and point. The researchers, with help of the advisors, have designed this crucial situation in four phases. Each of them aims at restoring the same figure (Figure 7), but the beginning and the tools are different for each phase. As tools, students always have a non-graduated ruler and an eraser, but the available templates change. The choice of the beginning and of the available templates is of course a didactical variable on which the teacher can act. In the proposed situation, from one phase to the other, the degree of freedom in positioning the templates to draw the figure increases and the perception of alignments is proving more and more critical for the success of the expected

[14] For instance, the ruler is a non-graduated ruler. To move lengths, students may use a compass or other informal instruments allowing to compare lengths without measuring them, like a paper strip with a straight edge or parts of the figure (here templates).

tracings. Figures 8 to 11 present four phases in tables, each of which has two sides where the left side of the arrow shows what is given to students and the right side shows the solution to complete the figure.

At a first glance, on the figure to restore, we can see two or three triangles with a common side lying on a quadrilateral, but to complete the figure, the students will have to see also two large overlapping triangles and certain relationships between segments and points in the model figure: for instance, some sides of the triangles and some vertices of the outer quadrilateral are aligned on the diagonals of this quadrilateral.

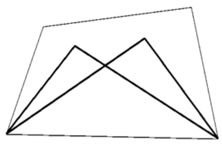
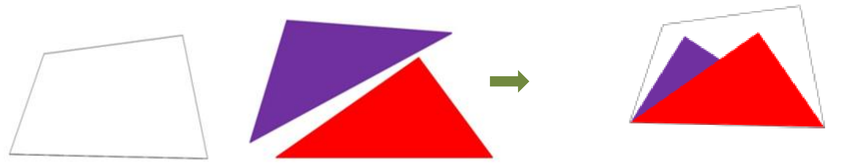


Figure 6

In Phase 1 (Figure 8), the beginning is the quadrilateral and the instruments are the (non graduated) ruler and two large triangles as templates. To restore the figure, the students must recognize them in the model (covering two triangles of the figure to reproduce) and place them on the beginning (the quadrilateral) to draw. The alignment of the sides of the two small triangles is given by the *milieu*: it is a consequence of the use of the templates since a side of the big triangle is the reunion of the two sides of the little triangles.



the beginning and the templates *how to restore*

Figure 7. Phase 1

In phase 2 (Figure 9), students have a "nibbled"[15] template T2, two corners of which are missing. To complete the figure (where the beginning includes a triangle T1), they have to know how to place the nibbled template: as it has no vertex, it is necessary to extend two sides of T1 with the ruler before placing T2 with two sides lying on the extension of those of T1. Thus the students have to use explicitly the alignments of the sides of T1 and T2.

[15] The template is "nibbled" to oblige students to extend the sides and not just make the summits coincide.

the beginning and the template *how to restore*

Figure 8. Phase 2

In phase 3 (Figure 10), the problem is to restore the quadrilateral from the two triangles and the beginning of two sides of the quadrilateral. There is no template, only a (non graduated) ruler. The sides of the quadrilateral and of the triangles have to be extended until the lines intersect to find the two missing vertices. It is necessary for the students to use "in action" the fact that we can get a point by intersection of lines.

the beginning and the templates *how to restore*

Figure 9. Phase 3

In Phase 4 (Figure 11), the beginning is the quadrilateral, the tools are two "nibbled" templates of T1 and T2. The problem requires students to see and to use the diagonals of the quadrilateral to place the templates before tracing. The templates were "nibbled" to entail the necessity to use the diagonals to place them. This phase may be seen as a reinvestment of the previous ones.

the beginning and the template *how to restore*

Figure 10. Phase 4

The concepts of TDS were used to elaborate the situation: the knowledge at stake was the notion of alignment (of points or segments) and intersection (of lines); the choice of didactical variables makes them necessary to solve the problems. We shall see in Section 2.2 how they can help the teacher to develop her/his practice.

2.2. Using TDS to develop teaching practices in the two contexts

In this part, we present our analyses using TDS concepts in relation with teaching practices and the way they can fit with certain professional reflections of teachers.

The case of multiplication

Devolution, informational jump and didactical contract

The possibility to draw ensured the devolution of the problem: this implicit model was available to all students. Task 1 did not explicitly need an arithmetic expression (eggs could be counted on the drawings). Nevertheless, the teacher aimed at such a representation for the students, and for that reason she changed the problem during the students' engagement with the task. She asked them to write an "arithmetic problem", referring by this question to the *didactical contract* associated with elementary word problems, which (for the students) involved translating them into "arithmetic problems". Another way to proceed might have been to make an *informational jump* by asking for, say, 150 portions.

Milieu, didactical variables

The objective situation (in Task 1) consisted of a person buying eggs for 12 portions of muffins, when each portion contains four eggs. The *material milieu* consisted of the eggs. The variables in the milieu that could be changed by the teacher are the numbers of portions and eggs. The knowledge supposed known was how to write an "arithmetic problem" representing a word problem.

Action, formulation, validation

Task 1 worked as an adidactical *situation of action* because the milieu was familiar enough for the students so that they could make an implicit model, in terms of drawings. After this, followed exchange of ideas in pairs, the purpose of which was sharing solutions and challenging each other when solutions were different. This did not work as intended. To a varying degree the students listened to each other, and there was no discussion when they had solved the task differently. Since there was no necessity to communicate to solve a task, this was not an adidactical *situation of formulation*. It would have been possible to have one by getting another student to use the explained method with a different number of portions, or with another recipe (with a different number of eggs). Another way to have a situation of formulation would have been to ask the students to agree on a method to apply it to a new question to come (before knowing the numbers).

Recapitulation of solutions at the end of Session 1 was focused on justification of students' methods, and hence it was a *situation of validation*. Because the necessity of validation came from the teacher, it was not an adidactical situation.

Institutionalization

During discussion of Task 1, a conflict occurred between the desired convention about the meaning of the factors in a product (understood as repeated addition), and the commutative property of multiplication as a mathematical operation. The teacher wanted the students to learn the convention that the first factor in a product signifies the number of equivalent groups and the second factor signifies the size of the groups. She used a large multiplier (1000) to motivate for the efficiency of product notation. But this created a conflict since it is easier to calculate 4·1000 than 1000·4 (two products with different senses but with the same reference).

The conflict created by the different commutative properties of the model and the situation in Task 1 was not resolved. The teacher continued on Task 2, where the commutativity of the situation (a rectangular array) was justified. After this, it would have been feasible to come back to Task 1 and say that it can be seen from Filipa and George's solution (Figure 4) that the eggs may be placed in a rectangular array and, as well, be calculated as 12+12+12+12. In this way, a connection might have been created between the two types of situations. This, further, would illuminate the relationship between the situation and the model—that the strength of numbers is to forget about the situation to do the calculations, then get back to the situation.

In summary, we used the concept of formulation to explain why the exchange in pairs was unsuccessful, and to suggest how this phase might be changed. Searching for a *purpose* of students' activity in adidactical situations may help the teacher to develop the (adidactical potential of the) milieu with which students interact. Further, the concept of institutionalization may help the teacher to understand the necessity of connecting students' productions (iconic representations of a non-commutative situation) with scholarly knowledge (commutativity of multiplication).

The case of geometry

We accompanied teachers in the preparation and implementation in class of the situation of geometry. Then we (researchers or advisors) conducted interviews with them. We use one of these interviews to illustrate the way in which some concepts of TDS may explain some difficulties encountered in class and be useful in the communication between teacher advisors and teachers and help the development of the observed teachers' practices.

Didactical contract, devolution

During the interview, one of the teacher advisors drew the attention of the teacher to the difficulties encountered during Phase 1 in the overlay of templates (she said:

"there was another problem, the superimposition, they [the students] refrained from superimposing… They prevent themselves from superimposing").

This problem is not due to a learning difficulty but to the constraints that the pupils give themselves. The concept of *didactical contract* can explain this difficulty and help the teacher to overcome it: the usual contract in geometry makes the students interpret the rules of the game that it is forbidden to overlap the templates. It was not cited by the advisor but it helps her advise the teacher: just allow students to overlap. Here the teacher must understand that this difficulty is not linked to a lack of mathematical knowledge from the student and that s/he needs clarify the rules of the game: tell the children that the templates may overlap; and that this clarification does not change the problem and the knowledge necessary to solve it: it relates to *devolution*. Taking support related to the concept of didactical contract, in this case, helps enrich the analysis of the teacher.

Milieu

In the designing of the situation, the evolution of the milieu (beginning and tools change among the phases) helps the students to change the way they look at the figure: from a vision of surfaces juxtaposed or overlapping to an analysis in terms of lines and points to construct (students' analysis is enriched through the tasks). During the interview, the teacher, in his commentaries about Phase 2, shows that he has understood that the changes in the milieu and the tasks asked of the students help them enrich their analysis or question their first analysis:

> "But me in the reflection of the kid, my interest precisely, it is that! We saw some things and when they get the templates, it exactly allows them to see what they have not seen! See, when B. tells me 'sir, the templates, they are not good', I answer 'ah yes, they are not good?!' So, once the kid knows where we could place the template, I can say 'well you see there is a triangle'."

During Phase 3, this teacher gives the students a string to help them locate the alignments, thus enriching the *milieu*. It is important for a teacher to understand that s/he can help students differently from intervening directly in the students' work. To provide the students with another instrument (templates, string) is a change in the milieu; it is another way to help the students without saying anything. TDS gives means to control the milieu in such a way that the students may learn, as much as possible, interacting with this milieu: TDS aims at characterizing situations (i.e. milieus) allowing students to learn some piece of knowledge by solving a problem, without significant help from the teacher.

Devolution, institutionalization

The teacher has to act so that the students solve the problem as their own, engaging their present knowledge and ready to acquire new knowledge. It is the *devolution* of the problem. In this case, the teacher, as well as some other teachers in our observations, chooses to begin with a phase of analysis of the figure. The difficulty in such a phase for the teacher is to let the students raise questions necessary to make precise the rules of the game, to postpone questions revealing in advance some crucial components of the figure or implying some construction. The interview shows that the observed teacher wants to give the students "good habits" and that, by "good habits", he refers to his own habits: "Myself, I begin like that: when I have a figure to reproduce, I look at it, I try to identify forms that I recognize, to find the links between them, to trace things that are not seen ...".

In fact, the proposed situation confronts the students with the resolution of a problem that makes the need for these "good habits" emerge from the students' reflections instead of being imposed or suggested from the beginning. These "habits" as well as some geometric knowledge linking the use of geometrical tools and geometrical concepts—such as "to set my ruler to draw a new line, I need two points or a segment already traced on the figure"—have to be formulated and pointed out for the students as something to know and use to construct geometrical figures. This corresponds to *institutionalization* in TDS. Clarifying the distinction between devolution and institutionalization helps the teacher develop her/his practice.

3. How TDS may help collaborations between researchers, teacher educators, and teachers in research on teacher development

In this part, we discuss how TDS intervene in the methodology of our researches, in particular we use the two contexts to examine how the collaboration between teachers, teacher educators and researchers might develop, focusing on the crucial question of links between the choice of the situation in relation to the knowledge at stake, devolution and institutionalization. The comparison of the two contexts informs on the specific contribution of TDS in understanding and developing mathematics teaching practices.

3.1. The case of multiplication

In the case of multiplication, questions in two arenas were identified: first, how to integrate a purpose—in the situation of formulation—so that the students would *need* the knowledge aimed at; second, how to solve a conflict—in institutionalization—between the situation to be modelled and a property of the mathematical model used to represent the situation. In collaboration between teachers and researchers, cases like the one analyzed here (with material from students' solutions and responses) may be used to discuss conditions and constraints (using TDS concepts) that enable or hinder students' opportunities to learn the knowledge at stake. This may then be used to modify and enrich the sequence for implementation in other classes.

TDS has been introduced to the LaUDiM project team by one of the researchers as a framework for investigating teaching and learning processes and for supporting didactical design in mathematics, where the particularity of the knowledge taught plays a significant role. After the project had been running for one year, the teacher (who has a Master's degree in mathematics education) was interviewed by one of the pedagogues about the significance of the project for her as a teacher of mathematics. The teacher expressed:

> "That is perhaps what I have learnt most from, I think, getting input from a somewhat different theory [TDS], a kind of model for teaching on the basis of which you can plan, which I had never heard of before".

From how TDS concepts have been used in the project, we understand that by *model for teaching* she means situations of action, formulation, validation, and institutionalization. Later, she said that defining the target knowledge was important: "[…] to choose exactly what [knowledge] we will work on is decisive for being able to design tasks that hit the goal". Further, the teacher commented on sequencing, that she had experienced how important it is to plan what (and why something) should come first in a teaching sequence. This was related to the importance of the pre-analysis, where the mathematical knowledge is analyzed by the team. The teacher claimed that being part of the project had clearly changed the way she thought about how a teacher should start a session on a mathematical topic. She is here seen to talk implicitly about *devolution*.

As part of data collection in the Norwegian project, researchers were asked to provide a written statement on potential impact of TDS on collaboration between researchers and teachers, regarding development of mathematics teaching. Two of the researchers focused on *institutionalization*, and this is what one of them wrote:

> "There is currently much focus on students' presentations of the work they have done in mathematics lessons. Very often this becomes show-and-tell, and some of the reason for this may be that teachers consider this part of the lesson mostly as a summary of what the students have been doing in the actual lesson. The concept of *institutionalization* may be useful to introduce to these teachers, so they can get a better understanding of what the teacher's role might (and should) be in this phase. [Institutionalization] to convey that the teacher has an important role in decontextualizing and helping students to put into words what kind of mathematics that has been worked on."

Even if knowledge of TDS concepts and models is shared among a group of researchers and teachers, there is a need for discussion of what the *target knowledge is* (or should be) in each case of designing a teaching sequence. This was pointed out directly by the teacher in the above extract from the interview, and indirectly by the researchers in their emphasis on the concept of institutionalization. Identifying the target knowledge requires pre-analysis and planning, preferably in a team of researcher(s) and teacher(s). This might not be realistic to carry out with teachers who are not part of a research project (i.e. if they have no reduction of teaching duties). However, analyzed teaching sequences (as the one on multiplication) can be adapted and implemented in other classes, for subsequent analysis. Even if TDS was pointed out as helpful by participants involved in the research reported here, more research is needed to know to what extent it is effective more broadly, for other teachers and researchers.

3.2. The case of geometry

In the case of geometry, the production of resources for regular teaching and teacher development proved to be a way to extend the collaboration between teachers, teacher advisors and researchers giving them a common aim. Our intention in this section is to present how this collaboration makes it possible to focus on the crucial question of links between the choice of the situation and the knowledge at stake, and explain how TDS concepts can be operationalized. We presented in Part 2 some examples showing how these concepts may help teachers interpret the choices made by the small group of researchers and advisors and develop their practices. Thus, on one side, notions arising from TDS can be mobilized by the teachers in action. On the other side, the concepts of TDS are explicitly present for the researchers at each stage of the process and help them interpret the teachers' questions and thus enrich their propositions.

The way this collaboration works is explained in Figure 12. In a first step (arrows n°1), researchers develop a situation based on research questions and hypotheses on the teaching and learning of geometry. TDS is the theoretical reference for the researchers exercising theoretical control on the analysis of knowledge, the definition of the situation, the milieu, the students' knowledge and the role of the teacher. But the theoretical control on the role of the teacher is to be tested and clarified especially in our case since we address all regular teachers. Therefore, in a second step (arrows n°2), this situation is discussed within the small group represented by the inner rectangle and a first document is written. At this stage, not everything can be anticipated by the small group who knows that difficulties will be brought to light during the implementation of the situation in class.

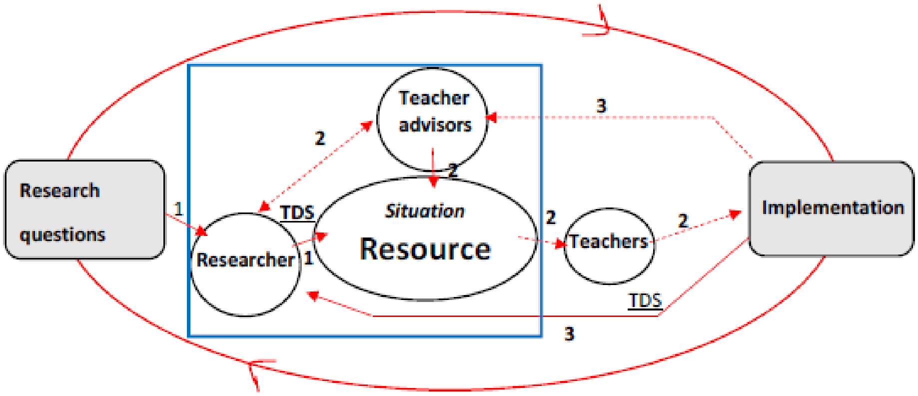


Figure 11. Use of TDS and collaboration between teachers, teacher advisors and researchers in case 2.

The situation is then presented to the teachers of the large group during a three-hour training session and a document (description of the situation and short guidelines for its implementation) is given to them. Implementation in class is accompanied by the advisors, observed (some of them with videos) by researchers or advisors, and followed by an interview. In a third step (arrow n°3), the small group analyzes the observations made and new questions emerge. Some of them give rise to pedagogical treatment but some of them require focusing on the crucial question of links between the choice of the situation and the knowledge at stake. These new questions enrich the work of the small group and the resource is modified. At each step, during the action itself or after the action, the researcher also takes information on the whole design process of the resource and analyzes how the different actors interact. The arrows are dashed when TDS is most often used implicitly (here, during the training) and the arrows are in solid lines when TDS is most often used explicitly (research). The outer arrows indicate the dialectic between research questions and observations.

We now give an example. In the initial document given to the teachers, there was no indication about the way to present the figure in class. When analyzing the class observations within the small group, we decided to take this issue into account and to give indications to the teachers (indeed, if the teacher develops a too precise analysis of the figure with the students, we see a risk of denaturalization of the situation). In a first time, the small group planned to draw up in the resource general advice essentially based on the question of devolution (explaining to teachers that students must understand what they have to do but should not be helped on how to do it before they try to reproduce the figure). Then, an advisor who is at the same time a teacher, while implementing the situation in her class, chose to write on the blackboard the first observations made by the students ("in this figure, I see ... a quadrilateral, two small triangles…”). Then, she hid this list and told the students that they would come back to it later. The other teachers observed did not write anything. This teacher advisor kept a track of the students' analysis in order to be able to complete it gradually with them. Giving a status to this writing, she initiated the process of institutionalization from the presentation of the figure.

This observation led the researchers to propose to the teachers to conduct a first analysis of the figure with students to complete it as the students’ research progress and return at the end. Thus, this observation helped the researchers to see how a more precise control of the role of the teacher could be implemented in the specific context of this situation. This example helped the explication in the small group of the way devolution and institutionalization are differently linked to the knowledge at stake and how this question might be taken into account in the resource for regular teachers. It is an example of the ways the collaboration between teachers, teacher educators and researchers is helpful: it helps researchers to see how concepts of TDS can be operationalized; it helps teachers or teacher educators working with researchers (in the small group) to explicitly approach the concepts; and it helps other teachers (using the resource) to gain some access to these concepts in the course of teaching.

3.3. Discussion

Comparison of the use of TDS in the two contexts

In both contexts, through the study of teaching, we have in perspective the study of the students’ learning and the teachers’ professional development—and our use of TDS is close one to the other. In both cases, the focus was on the design of the situation itself and its study. There are differences, however, in the objectives and research questions in the two contexts.

In the case of multiplication, the objective was to test the theoretical validity of the situation in relation to the essential elements about the target knowledge, whether the didactical intention was achieved or not, and why (i.e. to compare the *a priori* and the *a posteriori* analyses of the situation). Concepts of TDS have been made available to the teachers in order to give them tools for design and analysis of situations (arrows n°1 in Figure 13). As in the case of geometry, teachers implement situations (arrow n°2) and take part in the a posteriori analysis (arrows n°3). This explicit use of TDS concepts (arrows in solid lines) follows from the hypothesis that development of the teacher’s teaching practice is done through the implementation and analysis of a situation designed mainly by the teacher, based on *a priori* (epistemological and didactical) analyses done by the researchers and teacher in collaboration.

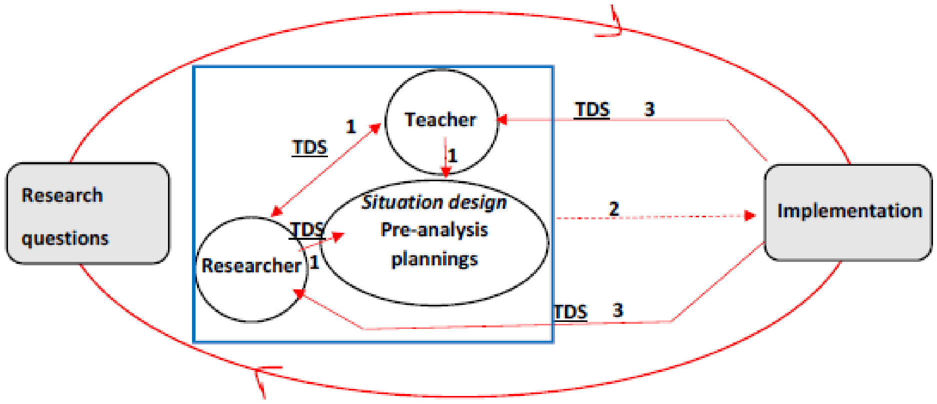


Figure 12. Use of TSD and collaboration between researchers and teachers in case 1

In the case of geometry too, the research questions comprise testing the theoretical validity of the situation in relation to the essential elements about the target knowledge, but they include also the study of the adaptability of this situation in regular education, taking into account the contributions of the teachers and the prospects of evolution of their practices. The objective was, after a first validation in the classes of teachers collaborating with researchers, to describe the situation in a resource with the perspective that regular teachers can use it without any direct interaction with the researchers or teachers collaborating with them. Therefore, the design process of the resource is at the center of the device (Figure 12) and not the situation itself as in Figure 13, and the TDS concepts were used only implicitly with the teachers (dotted arrows). This follows from the hypothesis that development of teachers' teaching practices is done through the implementation, analysis and adaptation of a situation first designed by the researchers.

Thus, the hypotheses and collaboration between teachers and researchers are different in the two cases. In the case of multiplication, the observations concern classes in which the teacher completed the design of the situation. In the case of geometry, except in one case, the observations concern classes in which the teacher did not take part in the design. The use of concepts of TDS is more explicit for the teacher in the case of multiplication than in the case of geometry. In the case of geometry, there is a big difference between the small group and the large group: in the small group, gradually, there is a certain familiarization, at least a use "in action" of the concepts of TDS, without expressing them, in the exchanges during the design of the situations and the analyses of class observations; in the large group the focus remains on decisions focused on practice.

Complementarity between TDS and other theoretical frames

In the two contexts, our research questions concern the teaching of a specific mathematical subject (multiplication or geometry) and the way to design situations acceptable by the teachers to improve their practice. The aim of a teaching situation designed according to TDS principles is students' development of meaningful, scholarly mathematical knowledge. Vygotsky's theory of concept formation is also about students' development of scholarly knowledge. Vygotsky (1934/1987) proposes that concept formation is the outcome of an interplay between spontaneous concepts and scientific concepts. However, as commented by Wertsch (1984), Vygotsky never specifies the *nature of instruction* of scientific concepts beyond general characteristics, in terms of teacher-student cooperation and assistance by the teacher, determined by the student's zone of proximal development (ZPD). On this point, TDS can be seen to complement Vygotskian theory in the way TDS provides tools for a fine-grained analysis of the progress of pieces of mathematical knowledge (from informal to formal mathematical knowledge), and what it takes for the teacher, in terms of designing a milieu and managing its evolution. For a discussion of compatibility of TDS and Vygotskian theory, see (Strømskag Måsøval, 2011, Chapter 2.7).

In the case of geometry, moreover, we wonder if an improving of teaching can result from taking ownership of a resource designed by researchers in collaboration with teachers and teacher advisors. We used TDS as a tool to design and analyze the implementation in classes of mathematics-teaching units, aiming at a generic and epistemic student's learning of some particular mathematical knowledge. The Double Approach (Robert and Rogalski, 2005) – rooted in Activity Theory (AT) – with its concept of proximities (cf. articles 2 and 3, in this volume) could be used to analyze the distance between what students do and know and the teacher's goals for the students, and how students' responses influence the actions and mediations of the teacher in trying to reduce this distance. However, there is an important difference in the nature of the didactical devices: whereas TDS aims at *adidactical* functioning of the knowledge, and its evolution, by designing and managing an appropriate milieu, the theory of proximities aims at *didactical* actions that the teacher can use to bridge the gap between students' existing knowledge and the new knowledge aimed at.

In comparison, TDS is a tool for the teacher and the researcher to determine *conditions* necessary for a situation to make a generic and epistemic student need the knowledge aimed at – here, the focus is on purpose and utility of the knowledge; the framework of proximities is a tool for the teacher to determine *actions in the course of teaching* or to prepare for this action, and for the researcher to analyze the teacher's actions, where the actual students' answers and questions have an impact on the teacher's decisions – here, the focus is on purpose and utility of the teacher's actions. We find the two theories complementary and potentially useful in combination to study mathematics teaching situations.

In the two research cases presented in the paper, we had questions about the knowledge itself, the means to make it accessible to students and the needs of a generic teacher. Of this reason we could not limit ourselves to the analysis of the teachers observed, and that is why we resorted to TDS.

Conclusion

We presented the use of TDS in a collaboration between researchers and teachers in two contexts in which research questions concern teacher education. We saw that TDS was helpful for researchers and teacher educators not only to design situations to learn some precise piece of knowledge but also to analyze what happens in class during the progress of the actual implementation of the situation and to identify questions useful to develop teachers' practices. In the two contexts, the analyses in terms of TDS were carried out by the researchers but, through some examples, we saw that they fit some professional questions from the teachers. These questions concern mainly their teaching goal, the way to organise some task for the students (related to the knowledge at stake) in such a way the students can know by themselves something about the pertinence of their answers, and the way to manage students' work. These questions correspond partly with the researcher's ones, but are more practical: The teacher must translate the concepts of TDS in terms of what s/he usually does to prepare or analyze her/his class.

The comparison of the two contexts raise a relevant question for the research: to what extent does the teacher need to know the concepts of TDS in a theoretical way (as concepts of a theory) to be able to use them in practice? Direct collaboration may help teachers develop their practices. However, it is neither realistic nor desirable to expect that all teachers can collaborate directly with researchers.

In the case of multiplication, TDS helps identifying questions concerning the milieu of the proposed situations and their adidactical potential, appropriate for the knowledge at stake. This in turn, makes it necessary to discuss the properties of the target knowledge. In the analyzed episode, a conflict occurred between a property of the target knowledge (the commutative property of multiplication) and one of the proposed situations aiming at multiplication as a model. Comparison of the *a priori* and *a posteriori* analyses of the sequence (which is an important part of TDS methodology) reveals shortcomings in the identification of the target knowledge (done in collaboration between the researchers and the teacher): the didactical intention (as expressed during planning) was related to the non-commutative situation (Task 1); the commutative situation (Task 2) was not part of the didactical intention.

In the case of the production of resources, TDS helps researchers and teacher educators to identify (through collaboration) questions concerning the way teachers interpret the design of situations using TDS, and how they enrich teaching from the implementation of such situations—particularly the choice of didactical variables. These new questions emerging from class observations lead to modification and enrichment of situations in the sense of TDS in such a way that regular teachers may more easily use them. Indeed, an important perspective is the question of the use of such a resource by teachers with no contact at all with research. For that, during the experimentation of the resource, it is necessary to understand the origin of the changes made to the proposed situation, and how the teachers take into account, throughout the implementation, the link between the situation and the target knowledge, how they react to what is happening in class to achieve the mathematical goals, and to the way knowledge can progress in class. To analyze teachers' point of view, from their professional practice, the Double Approach derived from Activity Theory is complementary to TDS, as commented above.

Even if design takes into account regular practices, important questions about the use of the concepts of TDS remain for researchers and teacher educators. First: how may this use be explained to other teachers using the resources, teachers who are not familiar with TDS concepts? Second: what teacher education should accompany such resources? The teachers need mathematical and didactical knowledge but, above all, they need to be able to put them into operation. That is why we, as researchers, consider that the concepts of TDS may remain implicit for the teacher, and focus our attention on how they operate (or not) in the teachers' practices. Nevertheless, we hypothesize that making them explicit is valuable for teacher educators accompanying the implementation in class of situations designed using TDS.

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CHRISTINE MANGIANTE-ORSOLA
LML, University of Artois
christine.mangiante@espe-lnf.fr

MARIE-JEANNE PERRIN-GLORIAN
LDAR, University of Artois
marie-jeanne.perrin@univ-paris-diderot.fr

HEIDI STRØMSKAG
Norwegian University of Science and Technology
heidi.stromskag@ntnu.no