ISABELLE BLOCH, PATRICK GIBEL

A MODEL TO ANALYZE THE COMPLEXITY OF CALCULUS KNOWLEDGE AT THE BEGINNING OF UNIVERSITY COURSE PRESENTATION AND EXAMPLES

Abstract. Our research focuses on the difficulties students encounter with the learning of calculus, considering that they have to cope with many more mathematical objects but also with new ways of reasoning – not only algebraic calculation, but also the practice of approximation, and a scaffolding way of using functions, limits, derivative, integrals, etc. to justify their answers. The semiotic facet of new objects, and the way to manage it, is also a source of great difficulties. In this article we establish that the model we built (Bloch & Gibel, 2011) is adequate to describe the work of University students who have to deal with the resolution of exercises about parametric curves and differential equations, even if this context is not an adidactical situation. In 2018, L2 students of Pau University were asked to solve little problems about limits, integral calculations or recurrence questions. They revealed difficulties to organize their knowledge and conclude about a limit, for instance. We give some examples of these troubles. We conclude for the necessity to implement adequate devices to help students better understand these 'new mathematics'.

Keywords. Calculus, students' understanding of mathematical signs and objects, reasoning processes, parametric curves, differential equations.

Résumé. Un modèle pour analyser la complexité de la connaissance du calcul au début de l'Université. Présentation et exemples. Notre recherche concerne les difficultés que rencontrent les étudiants dans l'apprentissage de l'analyse, en constatant qu'ils ont à faire face à de nouveaux objets mais aussi à de nouveaux modes de raisonnements : non seulement des calculs algébriques, mais aussi des approximations, et une articulation de l'usage des concepts comme les fonctions, limites, intégrales afin d'établir des preuves. Le statut sémiotique des objets est également une source de grandes difficultés. Nous montrons dans cet article que le modèle que nous avons décrit dans un article paru en 2011 est adéquat pour décrire les procédures et les erreurs d'étudiants de première année d'université dans la résolution de problèmes sur les courbes paramétriques et les équations différentielles. Nous concluons sur la nécessité d'introduire des dispositifs adaptés afin d'aider les étudiants à comprendre cette 'nouvelle' façon de faire des mathématiques.

Mots-clés. Analyse, signes et objets mathématiques et compréhension des étudiants, processus de raisonnement, courbes paramétriques, équations différentielles.

Contenu condensé de l'article

A l'université de Pau et des pays de l'Adour, une unité d'enseignement appelée '*Mathématiques du mouvement*' a été proposée en première année de licence ; elle tente d'articuler mathématiques et physique. Comment se passe cette articulation ? Permet-elle aux étudiants de mieux saisir les outils mathématiques en jeu ? Tenter

ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES, volume 24, p. 183 – 206.

de contextualiser les connaissances mathématiques dans des situations relatives à la cinématique permet-il aux étudiants de mieux comprendre les savoirs sous-jacents ?

En effet, à ce niveau certains concepts mathématiques subissent une transformation telle que leur statut et leur mode de fonctionnement s'en trouvent complètement bouleversés. Dans la transition secondaire/supérieur, ce décalage épistémologique est accentué par la configuration actuelle des contenus des programmes du secondaire : nous avons montré (Bloch, 2018) que les objets y sont étudiés de façon descriptive et non relativement à leur nécessité et leur rôle d'outil dans la construction de situations mathématiques ; c'est aussi le cas pour la fonction exponentielle par exemple, ou les équations différentielles. De plus, les procédures de résolution visées au supérieur sont beaucoup plus complexes et peuvent faire appel à des techniques de divers champs mathématiques ; les étudiants doivent identifier ces changements de cadres de façon autonome et savoir s'extraire du champ initial, ainsi que nous l'avions exposé dans Bloch & Gibel (2016). Nous rappelons que le modèle décrit dans Bloch & Gibel (2011) prend en compte trois axes d'analyse des productions étudiantes dans les situations : le niveau de milieu où se situe l'étudiant, la nature et les fonctions du raisonnement, et les signes mathématiques associés à ces productions, avec leur niveau d'abstraction, lequel est étroitement corrélé au niveau de milieu. Nous analysons les productions des étudiants relativement au modèle, et montrons que l'évocation du cadre cinématique ne suffit pas à aiguiller les étudiants sur une étude consistante des courbes paramétrées.

---

1. The learning of calculus: objects, signs and reasoning processes

Every researcher knows that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the University. The amount of research about this issue is now considerable and it goes on expanding and disseminating among communities, which is very good news (see for instance Tall, 1996, 2002; Reinholz, 2015; Bressoud, 2011; González-Martín, Bloch, Durand-Guerrier & Maschietto, 2014...). Studying this transition between the mathematical organisation in secondary school and University, in the teaching of (pre)calculus, we aim at classifying the different 'things' students have to cope with when they practise Calculus and move to Analysis. Our research focuses first on difficulties students encounter, considering that they have to manage many more mathematical objects *and* new ways of reasoning – not only algebraic calculation, but also approximation, and a scaffolding way of using functions, limits, derivative, integrals, etc... to justify their answers. Then we aim at building situations for a better learning of Calculus and Analysis.

1.1. The way to manage previous knowledge

The organisation in Secondary school takes into account some mathematical objects, such as functions, derivatives, integrals: but a number of researchers underline the fact that the way these objects are introduced leads to algebraic calculation and not to analytic work. For instance, students are supposed to calculate an integral but not to justify why it exists; to study the variations of functions with derivative, but not to have a knowledge about which functions get derivatives at which points, or not. So we can see that the *raison d'être* of a mathematical concept is not highlighted.
We notice that, even if teachers think of the structural level, in most cases they confront students only with the operational one. For instance, Ghedamsi (2015, p. 2109) analyses a first-year regular course at University and she concludes that:

> He (*the teacher*) does not intervene to enrich this work (*the students' work*) by emphasizing relationships among notions, by changing the setting of semiotic representations, by allowing openings to organize knowledge, by making assessments of knowledge, etc. (...) During the whole lesson, the students use methods from secondary school and do not succeed to shift to the use of methods expected at university level.

On the other hand, the old-style way of teaching (declaring mathematical knowledge in front of students) does not lead to success, so Artigue (2001) notices:

> ...an evidence of the limitations both of traditional teaching practices and of teaching practices which, reflecting the Bourbaki style, favoured formal and theoretical approaches. (Artigue, 2001, p. 208)

In this outlook, she agrees with Bressoud (2011) who identifies this traditional practice as “the worst way to teach”.

One effect at least is problematic: it leads students to ignore what mathematical concepts mean and aim at, and what we try to define and prove when we do mathematics (Bloch, 2015; Godino, 1996). We can observe that very often students are trying to calculate in a rather hazardous way, aiming to apply ‘in the dark’ rules they learned but in no way mastering concepts involved in the problem. Let us quote how Hubbard & West (1995) react about students' solutions of differential equations; it highlights how students can refer (or not) to (un)usual objects:

> The origin of this book was the comment made by a student: ‘This equation has no solution’. The equation in question did indeed have solutions, as an immediate consequence of the existence and uniqueness theorem, which the class had been studying the previous month. […] In fact, only very exceptional equations can be explicitly integrated. For instance, none of the following rather innocent differential equations can be solved by the standard methods: $x' = x^2 - t$; $x' = sin(tx)$; $x' = e^{tx}$. […] A proper attitude is the following: *differential equations define functions, and the objects of the theory is to develop methods for understanding (describing and computing) these functions.* (Preface, p. viii)

In the evaluation cases we study below, the problem seems not to be the way the notions have been taught: we got an access to the students' course notes, and they show relevant justifications and explanations. The didactical repertoire of the class has been elaborated by suitable exercises and situations, leading to highlight the operating mode of these concepts. But mathematical signs – and objects - are not always 'grasped' by students in the right way, a way that has suffered very important adjustments since secondary school. We can observe this complexification also in the answers students give to little problems we asked them (see 3.).

1.2. Mathematical objects and signs: complexification

At the beginning of University studies, students have to cope with functions, as in Secondary school, like rational ones such as polynomials, or sinus or cosine; they have to solve problems with exponentials, logarithms, but the derivatives can also generate new functions, and integrals too, or series: so objects may have a different status, and signs become polysemic. With respect to these signs, we notice that in Secondary school students operate frequently by implementing isolated techniques: in secondary mathematical context they can calculate in a rather straightforward way. But at University, they face complex signs and they have to associate different kinds of symbols, sometimes through a long proving process, for example to calculate a rather complex integral or to prove that a theorem is valid, which is not their responsibility in Secondary school. At University too, signs are multiform: for instance a derivative can be written *f'* but also *df/dx*; or *x* can be the function, so it will appear as *dx/dt*; a letter can nominate a variable, a function, or a parameter, which status is sometimes difficult to decode.

Moreover, the rules about the use of signs are imbricated, so if you try to calculate $\int cos^2x\ dx$ you have to linearize cos^2x because you cannot apply the rule of the primitive of x^2, just 'mixed' with the primitive of *cos x*, to cos^2x… and find $1/3.sin^3x$, as we saw once a student. This evolution of signs is even more evident considering the proving procedures within the calculus work: students have to understand and use new analytic methods, as it is well known, for limits with ε and α, and to master quantifiers, which reveals to be rather hard (see Gueudet, 2008; Chellougui & Kouki, 2013).

1.3. Reasoning processes

This new complexity requires that students adapt themselves to improve and perfect their reasoning processes: they must learn to use all the facets of knowledge and to adapt their "way of doing", taking into account all the aspects of a question and the requirements of the proofs. This implies that they become able to choose which knowledge is adequate to the problem, how to use it in a relevant way, and how to develop reasoning processes. This contrasts with the practices they get at Secondary school, where they usually just have to answer to punctual questions and are not accustomed to master the whole reasoning process.

They also have to deal with formal proofs and reasoning they have not been accustomed to understanding and managing. In Bloch & Ghedamsi (2004) we insist on the transition from an algorithmic work to complex techniques and technologies. Activities at tertiary level are centered on mathematical generalities, and their resolution requires the use of heuristic techniques: proving or conjecturing, or reasoning by *reductio ad absurdum*, or a research of counter-examples. These activities require a formalization work that is not usual at secondary level.

We can say that throughout these reasoning processes, signs (and then objects) work in a strong interaction, as seen in the example above: integrals with the primitive of sine and squares, but also techniques and technologies to prove. Among these technologies it is very important that students learn how to manage the new tools, such as quantifiers and the way to perform a valid reasoning up to its end.

This first approach leads us to some questions:

- How can we classify the objects, signs and reasoning processes students have to cope with during resolution of calculus problems?
- Which theoretical frame permits us to identify the different shapes and functions of students' reasoning activity? How (with which tools) is this activity likely to be observed?
- How is it possible to improve the way students can achieve an access to relevant concepts and methods for proving in Calculus, and which situations can be implemented at the transition between secondary and tertiary level?

2. Theoretical tools to analyze situations and students' productions: a model

Our main theoretical frame is TDS at University level (Theory of Didactical Situations, see González-Martín et al., 2014) but let us mention that the way we investigate is also correlated, in a way, to APOS theory, as we are concerned with existing *processes* in the way students do mathematics, and *objects* in the situations we try to organize, these objects being, at closing stages, expressed in schemas, that is, composite signs with various embodied rules.

Trigueros & Planell (2010) actually define the application of their theory within three stages: at the general *intra-stage* actions are possible, without identifying properties: it corresponds with M-2 in TDS (see below); the *inter-stage* is the one where students can identify relations between objects and processes, it is M-1 in TDS; and the construction of the knowledge occurs in the *trans-stage*, which can correspond to M0.

Trigueros & Planell (2010) insist on the importance of different theoretical frames to analyse phenomena from complementary perspectives. We also associate a semiotic analysis to TDS viewpoint (see Table 2 below): this allows us to consider the nature of the problems we ask students to solve, and evaluate which mathematical objects and representations they were able to apply in a right process and which ones they did not master.

Mathematics aims at the definition of 'useful' properties that can help to solve a problem and to better understand the nature of concepts. A strong characteristic of these properties is their invariance: they apply to wide fields of objects – numbers, functions, geometrical objects, and so on. This implies the necessity of flexibility of mathematical signs and significations. To grasp the generality and invariance of properties, students have to do many comparisons – and mathematical actions – between different objects in different notational systems. While the choice of pertinent symbols and different classes of mathematical objects is necessary to reach this aim, it is not sufficient: as we said, the situation in which students are immersed is essential. The Theory of Didactical Situations (Brousseau, 1997) claims that to make mathematical signs meaningful – which means that signs have a chance to be related to conceptual mathematics objects – it is necessary to organise situations that allow the students to engage with validation, that is, to work with mathematical formulation and statements.
As we said in González-Martín et al. (2014):

> In TDS, the fundamental object is the notion of *Situation*, which is defined as the ideal model of the system of relationships between students, a teacher, and a *milieu*. Students' learning is seen as the result of interactions taking place within such systems and is highly dependent on characteristics of these systems. […] Different phases of a *Situation* include: *action* (knowing appears as a means for action through models that can remain implicit), *formulation* (knowing develops through the building of an appropriate language), *validation* (knowing becomes part of a fully coherent body of knowledge). (p. 118)

The model of structuration of the didactical milieu used in our device is that of Bloch (2005). The chart below (Table 1) sums up the levels of milieu – from M1 to M-3.

M1 Didactical milieu	E1: reflexive student-subject	P1: Professor planner
M0 Learning milieu: institutionalization	E0: generic student-subject	P0: professor teaching
M-1 Reference milieu Formulation and validation	E-1: The student-subject as a learner	P-1: Professor Regulator
M-2 Heuristic milieu: action, research	E-2: The student-subject as an actor	P-2: Professor devolves and observes
M-3 Material milieu	E-3: epistemological student-subject	

Table 1. Structuration of the didactical milieu

The negative levels are of particular interest in the sequences we frequently study since they allow us to describe the emergence of a proof process in the setting up of an adidactical situation. The place where we hope to see the expected reasoning processes appear and develop is located at the articulation between the heuristic milieu and the reference milieu.

Let us specify however that, especially at the tertiary level, situations are not always entirely adidactical, but they can include an *adidactical dimension*, which allows letting students have a research activity and question the statements, even if they do not attain the final formal knowledge. The teacher must take this last stage in charge in M0. As said by Hersant & Perrin-Glorian (2005) and developed in Gravesen, Grønbaeck & Winsløw (2017), the milieu of situations - especially at the tertiary level - can show evidence of:

> a number of generic potentials for student learning, developed from and within the theory of didactic situations: adidactic potential, linkage potential, deepening potential and research potential. (p. 5)

It means that the milieu of such situations has been built in order to anticipate that students have a responsibility on knowledge: they are supposed to undertake reasoning, conjectures, calculations, decisions on a statement. A situation is:

> A system of relationships between students, a teacher, and a milieu (…) [where the milieu is] the set of material objects, knowledge available, and interactions with others, if any, that the learner has in the course of said activity. (González-Martín et al., 2014, p.119).

A link with previous knowledge, or related other knowledge, can also be shared in the situation; the knowledge can be deepened, and this deepening potential occurs mainly at the M0 level, that is, when the teacher institutionalizes the aimed knowledge; or the deepening can be organized later, in a further situation.

2.1. Identifying signs in situations

Considering the difficulties about how mathematical signs work and are likely to be understood, we thought it is necessary to introduce a semiotic dimension in our model. Our theorical reference is Peirce's semiotics. According to Saenz-Ludlow (2006):

> For Peirce, thought, sign, communication, and meaning-making are inherently connected. (…) Private meanings will be continuously modified and refined eventually to converge towards those conventional meanings already established in the community. (p. 187)

In Peirce's semiotics a whole sign is triadic and constituted by an *object* to which the interpretation refers, a 'material sign' (*representamen*), and an *interpretant*, the latter being an identity that can put the sign in relation with something – the object. A very important dimension in Peirce's semiotics is that interpretation is a *process*: it evolves through/by new signs, in a chain of interpretation and signs (representamen). The *interpretant* – the sign agent, utterer, mediator – modifies the sign according to his/her own interpretation. This dynamics of signs' production and interpretation plays a fundamental role in mathematics where a first signification has always to be re-arranged, re-thought, to fit with new and more complex objects – such as an integral for an example.

Peirce – who was himself a mathematician – organised signs in different categories; briefly said, signs are triadic, but they are also of three different kinds. We briefly sum up the complex system of Peirce's classification (ten categories, depending on the nature of each component of the sign, representamen, object, interpretant: see Everaert-Desmedt, 1990; Saenz-Ludlow, 2006) by saying that we will call an *icon* a sign referring to the object as itself – like a red object refers to a feeling of red. An *index* is a sign that refers to an object, as a proposition like: 'this apple is red'. A *symbol* is a sign that contains a rule. In mathematics all signs are symbols to be interpreted as *arguments* (see below), though they are not exactly of the same complexity; and so are the language arguments we use in mathematics for communication, reasoning, teaching and learning. Let us notice that signs can be either formal or linguistic: both will be taken into account. Are significant the arguments embodied in those signs.

About these different kinds of signs, Peirce wrote:

> First, an analysis of the essence of a sign, (stretching that word to its widest limits, as anything which, being determined by an object, determines an interpretation to determination, through it, by the same object), leads to a proof that every sign is determined by its object, either first, by partaking in the characters of the object, when I call the sign an Icon; secondly, by being really and in its individual existence connected with the individual object, when I call the sign an Index; thirdly, by more or less approximate certainty that it will be interpreted as denoting the object, in

> consequence of a habit (which term I use as including a natural disposition), when I call the sign a Symbol. (Peirce 1906, p. 495).

Let us specify that a sign as an Icon, for instance, could never give an argument, while an argument can very well be interpreted only as an Icon by somebody who would not get the right interpretant. This feature of Peirce's theory we find of course very relevant for mathematical signs interpretation. His theory is applicable to analyse the interpretation of mathematical signs, because all mathematical signs are *arguments*, even if of different levels, and because there will be some problems of misunderstanding if students do not interpret them in their suitable value.

Mathematical signs are symbols that give rise to arguments ('in consequence of a habit', here a mathematical habit of course) but of different levels: then '3' is the representamen of an argument because it always signifies that there are three elements somewhere (in a mathematics problem for instance; the relation between '3' and the number is a rule); but '123' (one hundred and twenty three) will be a more complex argument because the rule must include the decimal numeration, which is not the case in '3'. This complexity is what authorizes various interpretation of the same symbols, according to the mathematical competence of the interpretant.

This particularity of semiotic tools in mathematics will be in agreement with the relevant elements of the construction of a situation, because in a partially adidactical situation we must foresee the 'wrong' expression of mathematical properties, or the misunderstanding of some ostensives.

So Peirce's semiotics seems particularly appropriate for our research and will enable us to study more precisely the evolution and the transformations of the signs used by the different actors within the situation. In our application of this semiotics we use the three usual designations: icon, indexical sign and symbol-argument. Yet we do not consider the whole intricacy of Peirce's theory: it would be too complex to take into account and not necessary to interpret correctly students' actions in the situation. We just correlate *icons* with students' intuitions, drawings, examples, resolution attempts; *indexical signs* with local proofs, first tools for validation, more accurate reasoning, formulations of mathematical objects; and *symbols-arguments* with the concluding validation and formulation of the rules, and of the aimed knowledge – mostly under the teacher's responsibility. This categorization fits with the TDS model of milieu, which helps us to analyse students' productions within the two theories. We can also consider Peirce's idea of a 'hypoicon', which is a kind of schema involving a whole reasoning (see in Table 4 below).

We also want to take into account the semantic dimension – the *meaning* of the aimed knowledge – to analyse reasoning processes: this contributes to justify our choice of the TDS as a basis of our model. As we said, TDS organizes adidactical situations with three phases (corresponding to levels of the milieu): a heuristic one (students' action, oriented to the meaning of the aimed concept) grounded with a question; a formulation and validation phase; and a last one, institutionalization by the teacher. In this configuration the reasoning processes we take into account are as well valid or erroneous ones. This theoretical frame allows us also to develop an analysis of the *functions* of the reasoning processes within the situation (Gibel, 2015, 2018).

2.2. The didactical repertoire and the repertoire of representation

The work in the students' group leans first on the existing *repertoire*: all the semiotic means used by a teacher, and those he expects from his/her pupils through teaching, establish the didactical repertoire of the class – as defined by Gibel (2013). The didactical repertoire of the class can be identified as being part of the mathematical knowledge the teacher has chosen to explain, namely during validation and institutionalization phases of previous situations or previous lessons. The repertoire of representation is a constituent part of the didactical repertoire. It is made up of signs, diagrams, symbols and shapes and also linguistic elements (oral and/or written sentences), which make it possible to name the objects encountered and to formulate properties and results.

To sum up, in our model we consider signs, functions of reasoning, and levels of argumentation, while noticing that the reasoning processes elaborated by students and teacher during a lesson can occur in various ways: linguistic, calculative, scriptural, and graphic elements (see Bloch 2003). This leads to retain three main axes to study the reasoning processes.

- The first axis is linked to the nature of the situation: in a situation involving a research dimension, students produce reasoning processes which depend to a great extent on the involved phase of the situation, that is, the level of milieu (heuristic milieu, milieu of formulation or validation) (Table 1).
- The second axis is the analysis of the functions of reasoning. We aim at linking these two axes, showing how the reasoning functions are connected specifically to the levels of milieu and how these functions also *manifest* these levels of milieu.
- The third axis concerns noticeable signs and representations. These elements can be observed through different forms which affect the way the situation unfolds.

The schema below (Table 2) summarizes the construction of our model:

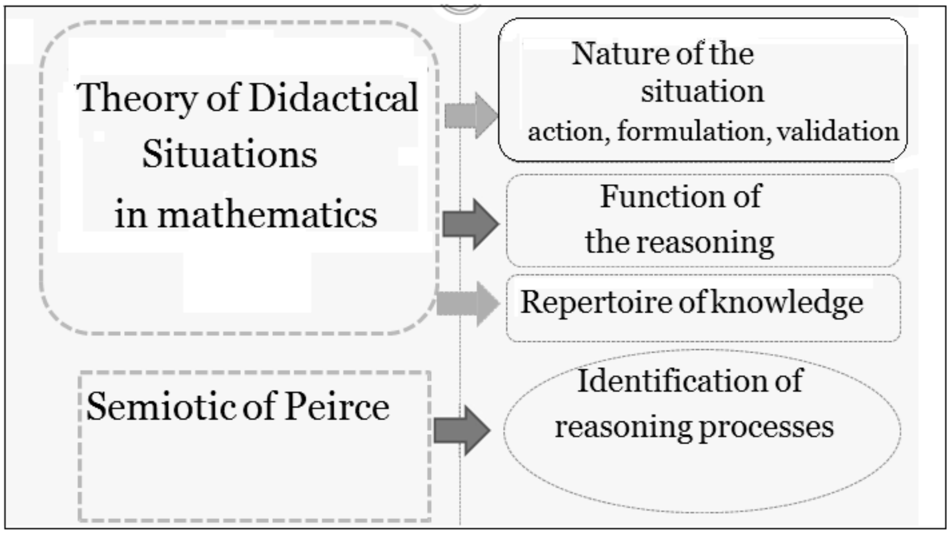

Table 2. Articulate TDS and Semiotic

Besides, we will observe if reasoning takes place on a semantic (SEM) or syntactic (SYNT) way. Durand-Guerrier (2010) defines precisely the semantic, syntactic and pragmatic dimensions by quoting Morris:

> semantics concerns the relation between signs and objects they refer to; syntax concerns the rules of integration of signs in a given system, and pragmatics the relationship between subjects and signs. (Morris, 1938)

The application of the model to a situation will then include an analysis of the milieu and semiotic analysis of the students and teacher's productions. We will interpret the conjectures, intuitions, signs and reasoning processes as an evolution of the didactical repertoire of the class, knowing that the situation aims at developing a mathematical knowledge in the field of calculus. It is necessary to see the signs students engage in the situation and which level of reasoning and rationality is involved in their work. The heuristic phase (students' action) must be grounded in a consistent mathematical question; then follows the formulation and validation phase; and institutionalization by the teacher. So for each level of milieu, we should be able to define the kind of reasoning (Table 3).

	M-2: research milieu	M-1: formulation	M0: institutionalization
Nature and functions of reasoning	SEM R1.1 - Intuitions on a drawing - Decision of a calculation - Exhibition of an example or counter example	SYNT/SEM R1.2 - Generic calculations and conjectures (right or wrong) - Decision on a math. objet	SYNT R1.3 - Formalization of proofs within the math. involved theory (mainly by the teacher)

Table 3. Functions of reasoning with regards to levels of milieu

This configuration provides tools to observe reasoning processes. The matrix notation R1.1, etc. is a tool to locate the level of attainment of the students' results. Then the second line of our model (Table 4) helps identifying the signs students use in the resolution of problems at stake in the situations, and the formal final signs. This leads us to:

	M-2 research milieu	M-1 formulation	M0 institutionalization
Level of use of signs and symbols	SYNT/SEM R2.1 Icons or indices depending on the context (first schemas, intuitions…)	SYNT/SEM R2.2 Local or more generic arguments: indices, calculations	SYNT R2.3 Formal and specific arguments: symbols *Hypoicons: schemas*

Table 4. Semiotic analysis of reasoning forms

We can see that the reasoning repertoire evolves within the situation: it leads us to provide a third line in the model (Table 5), with the aim of helping a teacher or a researcher to anticipate this evolution.

	M-2 research milieu	M-1 formulation	M0 institutionalization
Actualisation of the repertoire	SYNT/SEM R3.1 - Old knowledge - Enrichment at heuristic level: calculations, conjectures	SYNT/SEM R3.2 Enrichment at the argumental level: - statements, reasoning	SYNT R3.3 - Formalized proofs - Signs within the relevant theory - theoretical elements

Table 5. Evolution of the didactic repertoire and reasoning regarding the milieu

We could observe that within the problem resolution, different kinds of argumentation were proposed by students: so according to Lalaude-Labayle's work (2016), himself inspired by Peirce's categories, we add a dimension to our study; it concerns the nature of the proof students are able to propose, depending in which level of milieu they are located. This facet comforts the other tools we use to analyse students' productions.

Kind of argumentation	Abduction, induction, deduction	Induction, deduction	Deduction

Table 6. Argumentation

Let us give an example of each level of argumentation:

- Abduction: this situation is similar to another one I previously encountered, so I will choose this exponential function…
- Induction: this example involves these properties so I think it is an exponential function…
- Deduction: I could identify a derivative with all suitable properties, so it is necessarily an exponential function.

We want to underline the fact that an *a priori* analysis is necessary for each situation we choose to study: the model we built is efficient to perform this *a priori* analysis, as it allows us to anticipate resolution processes and difficulties. In this perspective, we aim at classifying reasoning, calculations, formulas, the nature of signs produced, and knowledge(s) likely to be expressed by students in the different phases, reflecting the situation in which students will be located.

The model can also be used to analyse 'ordinary' secondary or university teaching, as it allows detecting students' reasoning processes, use of symbols, and understanding of mathematical objects involved. We can analyse the students' templates while they try to solve a problem: they are first in a heuristic milieu M-2, trying to find a resolution process. Then they decide to undertake calculations, use of theorems, and their final action leads them to take position in a milieu M-1. In a different way, this can also be seen in the context of an evaluation, which we present in the next part.

3. Two situations of problem solving: how students cope with new knowledge and unusual signs

We made our model operate in classical adidactical situations (Bloch & Gibel, 2011), but we wanted to use it in quite 'ordinary' problem solving situations. Let us notice that these situations are partially adidactic because problems sent to students do not make use of trivialized knowledge.

In the model, we consider that the learning milieu M0 would be identified as the production of a standard solution expected by the teacher in terms of formulas and mathematical symbols. M-1 is a level of calculations that students undertake with their own formulas and conjectures. M-2, the heuristic milieu, contains students' attempts that highlight how they see the mathematical objects at stake. As said in Gravesen, Grønbaeck & Winsløw (2017), it is the place where students do A6 (in the authors' list):

> A6. Employ a non-formal "heuristic" representation of a mathematical object, to investigate it (p.5)

We then consider the productions of fourteen (14) students in this context – problem solving – at the University of Pau, in May 2014. The teaching unit involved is named: "Mathematics of the movement", which is interesting because a link is made between mathematical knowledge and physics problems; moreover, students can have access to the idea that a variable is not always denoted by *x*, it can be *t* as well, which refers to the time in a movement.

The milieu includes three exercises, the first one on polar and parametric functions, the second and the third ones on differential equations. Parametric curves and differential equations are especially interesting to study as they involve complex new signs, unusual processes for secondary students, and new kinds of reasoning. These reasoning encompass also mathematical objects, such as functions, limits, derivatives, but in a new way of thinking. As a number of authors, we can insist on:

> the importance of representations and mathematical visualization in the understanding of concepts. (Trigueros & Planell, 2010, p. 7)

3.1. Parametric curves

3.1.1. Proposed task

A parametric curve is of the type: $x = f(t), y = g(t)$. There are two functions x and y to study; students must understand that what is required finally is to describe the variations of *y* with respect to *x*, in the case of a movement for instance; so the study of the two functions *f* and *g* (including the calculation of their derivatives) is just a step (of R1.2 type) to interpret what happens with the curve of *y* while *x* being the final variable. Sketching the graph needs to give values to *t*, being sure that we got the 'whole' curve; or eliminating the parameter *t*, which may reveal to be complex, because in this case you cannot 'see' this number *t* varying. According to Trigueros and Planell (2010), we insist on the importance of the role of visualization in resolution of functions problems.

So, Weber and Thompson (2014) notice that understanding parametric functions suppose that one:

> … supports an image of scanning through values of one variable and tracking the value of another variable, (…so) imagine the quantities as coupled. (p. 77)

They also note that:

> It is nearly impossible for a novice student to conceive simultaneously a relationship between three quantities. (*ibid.*, p.74)

Another difficulty comes from the existence of tangents. In contrast to what happens with algebraic curves, parametric ones can have two tangents at the same point: this is a singular point that students did not meet before. They are expected to identify the nature of this singular point, for instance a cusp. They must first apply a formula ($x'(t)=0, y'(t)=0$) and then try to find the tangents at this point to be able to identify the nature of the singular point. Students have to engage a calculation and reasoning of successive derivatives that takes place at R2.3 or R3.2 level at least and involves specific interpretation about the objects at stake.

We classified the students' outcomes from S1 to S14. In May 2014 students were confronted to the following question:

Let us study the parametric curve defined by $x(t) = a\, t^2/(1+t^2)$, $y = a\, t^3/(1+t^2)$ *with* $t \in \mathbb{R}$. *Show that it is sufficient to study for* $t \geq 0$. *Determine the variations and confirm that the curve gets symmetry, an asymptote and a singularity.*

Students have to calculate $x(-t)$ and $y(-t)$ and conclude about the kind of symmetry; calculate the derivatives, build the variation table and do not forget the limits; and they must undertake pertinent interpretations of these results. The curve has a singularity, a cusp: they must find its coordinates and its nature. We expect that a difficulty can occur in the interpretation of derivatives: students are accustomed to calculating such derivatives but for algebraic functions one derivative is enough to find the variation of *f*. The asymptote can be a problem too, as $t \to +\infty$ when $x \to a$ and $y \to +\infty$. So the asymptote is vertical, but nevertheless when $t \to +\infty$, which can be a source of misunderstanding: for algebraic curves a limit where the variable tends to infinite corresponds to a horizontal asymptote.

Let us insist on the fact that these (little) problems involve a part of no trivial knowledge for students; this is why we say they are partially adidactical situations.

3.1.2. Analysis of the students' results

The first third writings show a global success (the notes are 19 or 20). Seven students are over 10, and the last seven productions are inadequate, the notes beeing from 8 to 2/20. So S11 is confused with the sign of *t* and *x, y*; half of the students have a problem with the symmetry and do not succeed with its identification; students make a lot of mistakes in signs, and in derivative calculation, though it has been studied in secondary school. The fourteen writings give the final result shown in Table 7, which confirms that the global success is not very high, and which shows a real contrast between the best students, who succeed in almost all the problem, and the last ones who do not master concepts and techniques.

Derivatives	***Exact Symmetry***	***Asymptote $x = a$***	***Singular point***	***Curve***
Correct calculation 12 (on 14)	7	Limites : 6 $x = a$: 4	Nul derivatives: 6 Nature of the point : 4	7 - according with the symmetry

Table 7. Students' difficulties

Student S1 does perfectly all that is expected: she calculates the derivatives, the behaviour of the function, draws the graph with the asymptote, and determines the cusp with its tangent, which needed to calculate $x^{(3)}(t)$ and $y^{(3)}(t)$ for $t=0$. S1 reaches the level R1.3, she makes a formalization of proofs within the required theory. Student S14 cannot do anything; five other students encounter difficulties to calculate derivatives, to interpret the symmetry, and to find the singular point. One student says that *a* should be the parameter. Another writes that the equation of the curve is $x(t)+y(t)$... : this student does not master the target concept of parametric curve, so he stays at level M-2 and tries heuristic (non-relevant) calculation.

So, we can see that even in M-2, some students do not appear to be able to undertake local adequate calculations, as they do not understand that they are no more in the case of a Cartesian function. There are errors about the nature of the asymptote, for instance: only six students calculate the limits and conclude about the asymptote, reaching the R2.2 level, but among these six, two of them write a wrong equation: $y=a$ instead of $x=a$. The students' productions also show calculation mistakes, especially in derivatives and primitives. The handling of singular points is not properly integrated: students are *unsettled* by the conditions for being a singularity, by the ways of finding the tangent... For instance, S6 tries to find the point by calculating $x=0$ and $y=0$ instead of their derivatives; S2, who succeeds in the exam, writes that: "every non collinear vector to the curve is tangent to the curve"…

Some students who calculate without mistakes encounter problems with the interpretation of their calculations: according to our model, we conclude that their use and interpretation of signs do not exceed the R2.1 or R2.2 level. To strengthen our approach, let us say that these students reveal to be able to calculate x'(t) and y'(t) using a well-known algorithm. However, they are unable to interpret the results because in the particular case of a singular point it is necessary to make complementary decisions on a mathematical object (R1.2 Table 3) such as calculate x''(t) and y''(t) and interpret them. Moreover, there is a need of explaining because you do that: the *raison d'être* of the calculation, that is, an "enrichment at the argumental level" as said in Table 5.

Those who succeed very well (four from the twelve) write sentences to explain that a singular point is given by *x'(t)=0, y'(t)=0*, applying a R2.2 or even R2.3 knowledge: they are producing statements with relevant signs that come under level R3.2 in Table 5.

One student says that it means that the speed is equal to zero; but only the first one S1 is able to calculate the tangent and identify the nature of the singularity, being clearly in the position R2.3 for all needed symbols: this seems to confirm that she is also located at level R3.3 with regard to Table 5.

3.2. Differential equations

In the second problem students had to cope with the solving of these two differential equations:

Exercise 3: Given the first order differential equation: $e^x yy' - x^2(y^2-9) = 0$
After separating the variable, solve the equation. Then solve the Bernoulli differential equation:

$$y' - \frac{4}{x}y - x\sqrt{y} = 0$$

3.2.1. A priori analysis

First, we consider the first order differential equation. Separating the variables implies preserving the initial shape, that is, not to develop the term $x^2(y^2-9)$, to obtain the following shape: $\frac{yy'}{y^2-9} = \frac{x^2}{e^x}$. This requires analysing preliminarily the features, the characteristics of the different mathematical signs appearing in the equation to anticipate the expected form. To solve this equation, students have then to transform y' as $y' = \frac{dy}{dx}$; then they can produce an algebraic form allowing them to integrate the terms.

Dealing with the term $\int \frac{x^2}{e^x}dx$ requires necessarily applying *twice* integration by parts. Considering the second part of the exercise, solving the Bernoulli equation gives rise to a number of difficulties: the first one consists in being able to make the substitution leading to the equation $2zz' - \frac{4}{x}z^2 = xz$. After simplification it can then be written: $2z' - \frac{4}{x}z = x$.

Students must solve first the homogeneous differential equation associated, and then they have to solve the inhomogeneous differential equation by variation of the constant, which can be a source of new difficulties. The technique of variation of the constant is a part of the new technical and technological tools of first year University course, so it is of Level R3.3 in our model.

3.2.2. Analysis of students' outcomes

First we analyze main difficulties encountered by students to solve the differential equation $e^x yy' - x^2(y^2\text{-}9) = 0$. The first one is to separate the variable to obtain $\frac{yy'}{y^2 - 9} = \frac{x^2}{e^x}$: among fourteen students only eight of them accomplished this task; for two of them this task was difficult and required several attempts as we expected. The next step of the resolution needs to compute $\int \frac{y}{y^2 - 9}dy$. Seven students out of eight were able to fulfil this task, but two of them represented the quotient as a sum of rational functions, because they did not acknowledge the derivative of the function $Ln(y^2 - 9)$. Recognize this primitive is of Level R2.2 because students have to identify a schema – a hypoicon according to Peirce – of different 'models' of derivatives/primitives, which variable is not always '*x*'. It supposes that the students' repertoire encompasses a lot of 'forms' that at this level they did not meet often enough. This informs us and is a good example of the lack of availability of mathematical tools to solve this problem. According to Table 5, these students are stuck at level R3.1. They try to operate with their previous repertoire and do not succeed in adapting this repertoire with the new tools studied.

We notice that only five students were able to deal with the term $\int \frac{x^2}{e^x}dx$ applying twice integration by parts. Then, only four students resolved this equation and obtained the whole solution.

As regards the Bernoulli equation, half of the students recognized an equation such as $y' + a(x)y = b(x)y^n$, with $n = \frac{1}{2}$ and $a(x) = -\frac{4}{x}$, $b(x) = x$. They were able to make the substitution $z = y^{\frac{1}{2}}$. But only five of them succeeded in obtaining

$2z' - \frac{4}{x} z = x$; two students did not dare to reduce the equation; they could not admit the possibility of dividing each term by z. Among these five students, the three other students implemented successful methods of solving.

We consider that this solution is not only procedural: students have to interpret that this integral includes well-known functions as x^2 and e^x, so they are supposed to know how to integrate and to derive them. Students must decide by induction that integration by parts would be an appropriate solution. We can presume that those who fail in this calculation are in an abduction process, trying to guess which tool would be likely to lead to blind success.

As a conclusion of this study, the model proves to be efficient to detect students' difficulties about signs, a new interpretation of functions and their derivatives in the case of parametric curves; troubles in the transformation of differential equations to obtain and be able to recognize interpretable 'shapes' as *y'/y*, for instance. Our model also allows– and this can be an important help for teachers and researchers – to become more precisely aware of the level students attain in their attempts to solve the problems: the fact that students made attempts to answer allows to investigate their knowing, their ability to choose mathematical tools and to make these tools work in a proper way in the problem. The elements of our model also allow a good analysis of the way students master the mathematical signs, considering that, as Trigueros and Planell (2010) notice, signs, visualization and understanding are strongly linked. We could add that a level for mastering calculation is essential too.

Conclusion

We can conclude that it is really difficult for students to get access to Level 3 of our model, although this level is the 'expert' one required: they frequently keep stuck at Level 1 with old non-adapted knowledge or false calculations (intra-stage according to Trigueros & Planell, 2010), or they try to work at Level 2 (inter-stage according to the same authors) but do not succeed in more complex calculations, especially when schemas (in Peirce's sense) are involved; or they make the expected calculation but are no more able to interpret it within the problem.

We notice that the involved activities, at this level, imply a very rich assortment of techniques, procedures, and a variety of occasions to apply formulas, which is not obvious even if students have understood the previous course. Yet the familiarity with this new field of knowledge is not easily established for a majority of students, and they are deficient in algebraic skills. Then the students' outcomes illustrate their numerous attempts to try to calculate and recognize well-known shapes within the heuristic milieu. We think that the difficulties highlighted in this study are not linked with the teacher's didactical choices, but they are common within the population of mathematical students, due to the reasons we evoked in the first part of this paper. Let us notice that the name of the course “Mathematics of the movement” is somewhere a fiction: these problems do not refer to a physics problematics, and even if it had been, we doubt it would have helped students to solve them, since difficulties reveal to come under understanding and calculating in mathematics. Maybe a strong reference to a physics phenomenon would have been more a source of additional difficulties.

We also want to point out the missing knowledge in the (French) secondary curriculum: students study no more the composition of functions. Yet they must recognize the kind of schemas we see in a differential equation as above, and this implies to detect which functions are at stake and how they appear in the formula. As students have no familiarity with 'the whole formula' they try to interpret each element separately, which has no meaning. So, most of them do not achieve the level R3.

We could also formulate these obstacles by saying that students fail in doing a pertinent association between syntactic and semantic methods: they are stressed with calculations and cannot control the meaning of the operations they have done.

These results show that it would be necessary to introduce relevant situations with an adidactical dimension aiming at the introduction of the Calculus concepts. Notice that building such situations for complex mathematical knowledge is not always easy. Moreover, the implementation supposes that the teacher masters the course of the situation and the students' progress: this implies specific professional skills, the lesser being to be open minded to students' productions, even if 'false', and able to provide some help but not the whole solution.

The next step of our work should be to find relevant situations for the teaching of Calculus, both in an introductory way in Secondary school, and at University.

Bibliography

ARTIGUE, M. (2001). What can we learn from Educational Research at University level? In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 207—220). New York: Springer.

BLOCH, I. (2003). Teaching functions in a graphic milieu: what forms of knowledge enable students to conjecture and prove? *Educational Studies in Mathematics*, **52**, 3-28.

BLOCH, I. (2005). *Quelques apports de la théorie des situations à la didactique des mathématiques dans l'enseignement secondaire et supérieur.* Note de synthèse de l’Habilitation à Diriger les Recherches. Université Paris 7. Available on: https://tel.archives-ouvertes.fr/tel-00012153

BLOCH I. (2015). Concepts, objets, symboles, enseignement des mathématiques : Quelques réflexions sur l'épistémologie et la didactique. *Petit x*, **97**, 71-79.

BLOCH I. (2018). Connaissances sur les nombres des élèves de fin de secondaire et adaptation à l'université. *Petit x*, 106, 65-77.

BLOCH, I., GHEDAMSI, I. (2004). The teaching of calculus at the transition between Upper Secondary School and University. *Communication to the Topic Study Group 12, ICME 10.* Copenhagen, Denmark. Retrieved from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.123.6490&rep=rep1&type=pdf

BLOCH, I., GIBEL, P. (2011). Un modèle d'analyse des raisonnements dans les situations didactiques : étude des niveaux de preuves dans une situation d'enseignement de la notion de limite. *Recherches en Didactique des Mathématiques*, **31-2**, 191-227.

BLOCH, I., GIBEL, P. (2016). A model to analyse the complexity of calculus knowledge at the beginning of University course. In E. Nardi, C. Winsløw & T. Hausberger (Eds.), *Proceedings of INDRUM 2016, First Conference of the International Network For Didactic Research in University Mathematics* (pp. 43-52). Montpellier: Université de Montpellier and INDRUM.

BRESSOUD, M. (2011). Historical Reflections on Teaching the Fundamental Theorem of Integral Calculus. *The American Mathematical Monthly*, **118-2**, 99-115.

BROUSSEAU, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer.

CHELLOUGUI, F., & KOUKI, R. (2013). Use of Formalism in Mathematical Activity Case Study: The Concept of Continuity in Higher Education. *Proceedings of the Eighth Conference of the European Society for Research in Mathematics Education, CERME 8,* Antalya, Turquey. Retrieved from: https://www.mathematik.uni-dortmund.de/~erme/index.php?slab=proceedings

DURAND-GUERRIER, V. (2010). Semantic perspective in mathematics education. A model on theoretic point of view. *Proceedings of ICME 11*, Mexico. Retrieved from https://www.mathunion.org/fileadmin/ICMI/files/Publications/ICME_proceedings/ICMI_2011/Durand-Guarrier.pdf

EVERAERT-DESMEDT, N. (1990). *Le processus interprétatif : introduction à la sémiotique de CS Peirce.* Liège : Mardaga.

GHEDAMSI, I. (2015). Teacher management of learning calculus: case of sequences at the first-year University. In Krainer, K., Vondrová, N. (Eds.). *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)*. (p. 2103-2109). Prague, Czech Republic: Charles University. Retrieved from: http://www.erme.tu-dortmund.de/~erme/CERME9/index.php

GIBEL, P. (2013). Presentation and setting up of a model of analysis for levels of proof in mathematics lessons in primary school. In Ubuz, B., Haser, C., Mariotti, M. A. (Eds.). *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education (CERME 8)*. (p. 127-135); Ankara, Turkey Retrieved from http://www.erme.tu-dortmund.de/~erme/CERME8/index.php

GIBEL, P. (2015). Mise en œuvre d'un modèle d'analyse des raisonnements en classe de mathématiques à l'école primaire. *Éducation et Didactique*, **9-2**, 51-72.

GIBEL, P. (2018). *Elaboration et usages d'un modèle multidimensionnel d'analyse des raisonnements en classe de mathématiques*. Note de synthèse de l'Habilitation à Diriger les Recherches, Université de Pau et des Pays de l'Adour. Available on: https://hal.archives-ouvertes.fr/tel-01919188

GODINO, J. (1996). Mathematical concepts, their meanings, and understanding. In L. Puig & A. Gutierrez (Eds.), *Proceedings of XX Conference of the International Group for the Psychology of Mathematics Education* (vol.2, pp. 417-425). Valencia: Universidad de Valencia.

GONZÁLEZ-MARTÍN, A., BLOCH, I., DURAND-GUERRIER, V., MASCHIETTO, M. (2014). Didactic Situations and Didactical Engineering in University mathematics: cases from the study of Calculus and proof. *International Journal of Research in Mathematics Education*, **16(2),** 117-134.

GUEUDET, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics* **67**, 237-254.

GRAVESEN, K., GRØNBAECK, N., WINSLØW, C. (2017). Task Design for Students' Work with Basic Theory in Analysis: The Cases of Multidimensional Differentiability and Curve Integrals. *International Journal for Research in Undergraduate Mathematics Education.* N°**3**. p. 9-33

HERSANT, M., PERRIN-GLORIAN, M.-J. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactic situations. *Educational Studies in Mathematics*, **59(1–3),** 113-151.

HUBBARD, J.H., WEST, B.H. (1995). *Differential Equations: A Dynamical Systems Approach.* New York: Springer.

LALAUDE-LABAYLE, M. (2016). *L'enseignement de l'algèbre au niveau universitaire. Etude épistémologique et didactique*. Thèse de doctorat, Université de Pau et des Pays de l'Adour.

MORRIS, C.W. (1938). *Foundations of the Theory of Signs*, Chicago: University of Chicago Press Cambridge University Press.

PEIRCE, C. S. (1906) Prolegomena to an Apology for Pragmaticism. *The Monist.* Vol. 16, 492-546

REINHOLZ, D. (2015) Peer-Assisted Reflection: A Design-Based Intervention for Improving Success in Calculus. *International Journal of Research in Undergraduate Mathematics Education*, Volume 1, Issue 2, pp 234–267

SAENZ-LUDLOW, A. (2006). Classroom interpreting games with an illustration. *Educational Studies in Mathematics*, **61**, 183-218.

TALL, D. (1996). Function and calculus**.** In Bishop et al. (Eds) *International Handbook of Mathematics Education*, (p. 289-325). The Netherlands: Kluwer.

TALL, D. (2002). Natural and formal infinities. *Educational Studies in Mathematics*, **48(2-3)**, 199-238.

TRIGUEROS, M., MARTINEZ-PLANELL, R. (2010). Geometrical representations in the learning of two-variables functions. *Educational Studies in Mathematics*, **73(1),** 3-19.

WALL, C.T.C. (2004). *Singular points of plane curves*. Cambridge University Press. Available on: https://www.cambridge.org/core/books/singular-points-of-plane-curves/C805A76E40F5DB806A0F2FC6CBEA63A0#

WEBER, E., THOMPSON, P.W. (2014). Students' images of two variables functions and their graphs. *Educational Studies in Mathematics*, **87,** 67-85.

ISABELLE BLOCH
Université de Bordeaux, France
isabelle.bloch@u-bordeaux.fr

PATRICK GIBEL
Université de Bordeaux, France
patrick.gibel@u-bordeaux.fr