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VISUAL ARTS IN U.S. GEOMETRY TEXTBOOKS ALIGNED WITH THE COMMON CORE STANDARDS

Abstract. This study investigates visual arts references in five U.S. high school geometry textbooks aligned with the Common Core State Standards for Mathematics. In all of the textbooks, architecture is the most commonly used context. More than half of the visual arts references are in the exercises. Congruence is the domain most often used, followed by Similarity, Right Triangles & Trigonometry. The visual arts references support the four traditional arguments justifying the geometry course but mostly support the goal of teaching geometry in ways that allow students to draw upon their intuition.

Résumé. Les arts visuels ajustés aux standards dans les manuels de géométrie étatsuniens. Cette étude examine les références aux arts visuels dans cinq manuels de géométrie du secondaire en se référant aux normes du tronc commun de mathématiques. Dans tous les manuels, l'architecture est le contexte le plus couramment utilisé. Plus de la moitié des références aux arts visuels se trouvent dans les exercices. La congruence est le domaine le plus souvent utilisé, suivi par la similitude, les triangles rectangles et la trigonométrie. Les références aux arts visuels soutiennent les quatre arguments traditionnels justifiant le cours de géométrie, mais soutiennent surtout l'objectif d'enseigner la géométrie de manière à permettre aux étudiants de faire appel à leur intuition.

Keywords. Mathematics, curriculum, geometry, visual arts, standards.

Mathematics and art have been connected throughout history. Problems such as how to tile a flat surface intrigued the mathematician Roger Penrose, whose name is associated with Penrose tilings (Livio, 2002). The Dutch artist, Maurits Cornelis Escher, had communications with mathematicians, including George Pólya, Harold Coxeter, and Roger Penrose; used mathematics as an inspiration for his art; and used art to showcase important mathematical ideas (Schattschneider, 2010). There are many historical examples where mathematics and art converge, such as the development of projective geometry in relation to perspective drawing and the use of the golden ratio in architecture (Pedoe, 1976). Researchers on ethnomathematics have also unpacked the mathematical work involved in various practices such as weaving baskets and making pottery designs (Ascher, 1991). The connections between mathematics and art have the potential to provide a context for the study of mathematics in school. Most recently, the National Council of Teachers of Mathematics (NCTM) proposed that the high school mathematics curriculum should provide opportunities for students to appreciate mathematics: "High school

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mathematics can potentially cultivate in students a sense of wonder, beauty, and joy—and doing so is an important but often neglected purpose for teaching mathematics" (NCTM, 2018). This call broadens prior perspectives regarding the goals for school mathematics stated in the *Principles and Standards for School Mathematics* (NCTM, 2000). At the same time, this new goal opens opportunities to integrate visual arts and the mathematics curriculum. Visual arts provide opportunities to develop an aesthetic sense that can be a source for appreciating mathematics. By visual arts, we refer to "a visual object or experience consciously created through an expression of skill or imagination" (Encyclopædia Brittannica, n.d.).

In this study, we aim to examine whether and how the current high school mathematics curriculum establishes connections with visual arts. We focus on the U.S. high school geometry curriculum. The high school geometry course has been constantly offered in the U.S. since the 1840s (Quast, 1968), despite various attempts at integrating the mathematics curriculum (Stanic & Kilpatrick, 1992). A historical analysis of the justifications given for the geometry course in the 20th century demonstrates the coexistence of various discourses supporting why students should have to learn geometry (González & Herbst, 2006). For example, advocates of the geometry course stated as reasons that the course could prepare students for the workforce, instill in students an appreciation for geometric patterns, teach students how to generate mathematical conjectures, and engage students in applying logical reasoning to ordinary situations. These curricular expectations are at times contradictory and create tensions when teachers are trying to fulfill various demands.

Through our analysis of references to visual arts contexts in geometry textbooks, we examine connections between the arguments justifying the geometry course in the 20th century and opportunities for achieving the new goal of appreciating mathematics proposed in recent NCTM documents. Mathematics education research has profited from textbook analyses to better understand students' learning opportunities (e.g., Dimmel & Herbst, 2015; Herbst, 2002; Hunte, 2018; Mesa, 2004; Otten, Gilbertson, Males, & Clark, 2014; Thompson, Senk, & Johnson, 2012). We study textbooks with the goal of understanding the transition between the intended curriculum in the standards and the written curriculum (Herbel-Eisenmann, 2007). Geometry teachers use textbooks as a resource in their classrooms in various ways, for example, emphasizing tasks that require explanation, avoiding challenging tasks, or supplementing the textbook with other resources (Sears & Chávez, 2014). Consequently, an examination of geometry textbooks can provide a starting point for understanding teachers' uses of curricular materials. To our knowledge, there has not been an analysis of visual art references in geometry textbooks. Our intention is to start a conversation about the possibility of implementing a mathematics curriculum that uses visual arts as a realistic context aligned with Freudenthal's goal of experiencing "mathematics as a human activity" (Gravemeijer & Terwel, 2000, p. 780). Visual arts can be an entry point to engaging in mathematical activity, thereby broadening students' appreciation for mathematics.

1. Theoretical Underpinnings

We draw upon two perspectives to guide this study. First, we review prior work on the geometry curriculum that establishes various discourses that justified the existence of the American geometry course in the 20th century (González & Herbst, 2006). We illustrate this framework with examples from policy documents that refer to visual arts when establishing the goals of teaching geometry and use the framework to examine textbooks aligned with the current Common Core State Standards for Mathematics (CCSSM), which have been implemented in most states in the U.S. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Second, we discuss the Realistic Mathematics Education (RME) theory in relation to curriculum theory (Freudenthal, 1971, 1991; Gravemeijer & Doorman, 1999; Gravemeijer & Terwel, 2000; van Den Heuvel-Panhuizen, 2003, 2005). These two perspectives frame our examination of visual arts references in mainstream U.S. geometry textbooks as a case of how the curriculum sets expectations for mathematics students.

1.2. Justifications for the Geometry Course

The geometry course holds a special place in the U.S. mathematics curriculum. The geometry course has survived attempts to be dissolved and integrated with other mathematics courses (Stanic & Kilpatrick, 1992). An analysis of important texts discussing the geometry curriculum in the twentieth century yields the coexistence of four distinct discourses justifying the geometry course (González & Herbst, 2006). The term *discourse* refers to the following definition:

> We shall call discourse a group of statements in so far as they belong to the same discursive formation; it does not form a rhetorical or formal unity, endlessly repeatable, whose appearance or use in history might be indicated (and, if necessary, explained); it is made up of a limited number of statements for which a group of conditions of existence can be defined. (Foucault, 1972, p. 117)

The four discourses justifying the geometry course are a *mathematical argument*, an *intuitive argument*, a *formal argument*, and a *utilitarian argument*. The mathematical argument justifies the existence of the geometry course as an opportunity for students to engage in making and proving mathematical conjectures, analogous to mathematicians' work. Proponents of this argument establish that geometry is a domain where students can learn to appreciate an axiomatic system. As a result, geometry curricula that are aligned with this discourse support students in identifying differences between postulates, definitions, and theorems, as well as engaging students in crafting mathematical proofs. The intuitive argument builds on the idea that geometry is particularly connected with our experiences in the world. Students' opportunities to appreciate geometry through activities such as visualization can help them to use their intuition in developing mathematical knowledge. Proponents of this argument also state that geometric concepts can provide a context for applying the properties of algebra. Geometry curricula aligned with the intuitive argument make geometric knowledge accessible to students with various learning needs. One example of such a curriculum is the informal geometry course intended to increase opportunities for students to achieve geometric literacy (Cox, 1985). Another example is Serra's (1997) geometry textbook, which provides examples of geometric designs in various artifacts and hands-on activities such as paper folding and drawing. The formal argument stems from the notion that geometry students can study logical reasoning that is transferable to everyday situations. From this perspective, the geometry course can teach students about participating in a democratic society (Fawcett, 1935, 1938). The utilitarian argument emphasizes that the geometry course can prepare students for future jobs. In doing so, the geometry curriculum provides examples of ways in which geometry is applicable in the workforce. While some geometry curricula may be more or less aligned with one of the arguments, the arguments coexist within curricular materials. As a result, curricular materials reveal tensions in the justifications for the geometry course. In our analysis of the references to visual arts in the current geometry curriculum, we seek to understand the relationships among the four discourses that have justified the geometry course in the past.

1.1.1. Visual Arts in Reports by Curriculum Committees

Two committees established fundamental perspectives on the study of geometry. The report of the Committee of Ten addressed the high school curriculum (Eliot, 1893/1969), whereas the report of the Committee of Fifteen specifically addressed the geometry curriculum (Slaught et al., 1912). The justifications that these reports established for the study of geometry continue to be used today (González & Herbst, 2006). The report of the Committee of Ten, published in 1893 and led by Harvard University President Charles W. Eliot, promoted humanistic ideals and set high expectations for high school students (Kliebard, 1995). The report of the Committee of Ten included sections on two types of geometry: concrete geometry and formal geometry. Concrete geometry was intended for 10-year-old children and consisted of providing opportunities for students to develop an understanding of geometric figures through drawing and work with physical models. There are no references to visual arts in the concrete geometry section. The section on demonstrative geometry highlights the importance of using logical reasoning and engaging students in crafting mathematical proofs. There is only one mention of possible connections with visual arts: "A place should also be found either in the school or college course for at least the elements of the modern synthetic or projective geometry" (Eliot, 1893/1969, p. 116). Nevertheless, projective geometry has typically been excluded from the high school curriculum except for optional explorations.

The report of the Committee of Fifteen on the Geometry Syllabus (Slaught et al., 1912) provides more details about the geometry curriculum since it had a particular focus on geometry. The first reference to visual arts in relation to the high school geometry course concerns scale drawings: "The usefulness and the necessity of the operation should be emphasized, and such application as the drawing of house plans, the copying of patterns on a smaller scale, etc., should be given" (Slaught et al., 1912, p. 91). In a section of the report entitled, "sources of problems," the authors list "Architecture, Decoration, and Design" (Slaught et al., 1912, p. 96) and provide a classification of problems from this domain in relation to the construction of figures, proofs of geometric properties, and computations. The authors mention that architectural ornaments (e.g., tilings and windows) can provide sources for problems. They also include sample problems using these contexts, including an Arabic design for floors and a church window design. The authors reproduce the geometric constructions that produce the designs, identifying specific properties of the geometric figures (such as symmetry), making statements about the locus of particular points, and providing formulas describing the geometric figures constructed. The 2-page analysis of geometric designs in architecture and design showcases visual arts as a source for geometric problem-solving. The authors of the report include a bibliography of 26 books with sources for geometry problems. Nine of those books pertain to visual arts, including Mabel Sykes' (1912) book about geometry in art and decoration[1]. These examples have the dual purpose of preparing students for the workforce as designers and architects while also using architecture and design to teach properties of geometric objects. In these references, the authors of the Committee of Fifteen align the purpose of the geometry course with the utilitarian and mathematical arguments.

1.1.2. References to Visual Arts in the NCTM Standards

Most recently, two NCTM documents have made specific statements about the teaching of geometry. The 1989 Standards include two geometry-specific standards (7 and 8): "Geometry from a synthetic perspective" and "Geometry from an algebraic perspective" (NCTM, 1989, pp. 157–162). The references to visual arts are only in Standard 7. In the "focus" section, standard 7 mentions the arts when showing examples of various applications of geometry and gives the example of perspective drawing. In the discussion, there are two examples of visual arts. The

[1] Mabel Sykes was a member of the Committee of Fifteen.

authors state, “Geometry also provides an opportunity for students to experience the creative interplay between mathematics and art” (NCTM, 1989, p. 158). One example is tiling: finding regular polygons that can tile the plane to make regular and semiregular tiling patterns as well as irregular polygons that can tile the plane. The example ends with references to Escher’s work, stating, “The last activity is appealing to many high school students and provides an excellent setting for creative expression” (NCTM, 1989, p. 158). A sample Escher-like drawing is also included. These references are aligned with the mathematical and intuitive arguments. The question about what polygons can tile the plane is a mathematical one that requires students to make and test conjectures. In the discussion, the authors of the 1989 Standard identify various geometric concepts that can support students’ discovery, such as their knowledge about the “angle-sum property of a triangle” (NCTM, 1989, p. 158). At the same time, the notion that students can show creativity through the creation of Escher-like tessellations is aligned with the intuitive argument because it allows students to engage in mathematics through art. The second visual arts reference in the standard is in its mention of careers in which visualizing three-dimensional figures is important, such as architecture, thus aligning the references to the utilitarian argument.

The *Principles and Standards for School Mathematics* (NCTM, 2000) refers to connections between the arts and geometry when discussing the Geometry Standard for Grades 9–12: “Geometric ideas can be useful both in other areas of mathematics and in applied settings. For example, symmetry can be useful in looking at functions; it also figures heavily in the arts, in design, and in the sciences” (NCTM, 2000, p. 309). The Standard includes an example of perspective drawing to illustrate how geometric ideas support the procedures that artists apply in representing equidistant parallel lines. Using the context of drawing three telephone poles in one-point perspective, the authors provide the step-by-step procedure for construction using mathematical terms to identify parts of the diagram (e.g., midpoint, diagonals, rectangle) and the term “vanishing point,” which is used in art and in mathematics. This example highlights a topic that historically has connected mathematics and art: perspective drawing. At the same time, the reference to perspective drawing is in the context of the work of artists. That is, students who learn how to apply geometric properties to perspective drawings could apply this knowledge to their work as artists. This example in the Principles and Standards illustrates how the study of geometric problems situated in a visual arts context can support the utilitarian argument.

1.2. Problem Contexts and the Realistic Mathematics Education Theory

The RME theory was originally proposed by the Dutch mathematician and mathematics educator Hans Freudenthal and further developed in mathematics education (Gravemeijer & Terwel, 2000). The theory is based upon the assumption that mathematics is a human activity and, therefore, that it is important for students to learn mathematics through guided reinvention. Freudenthal (1968) stated, “What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics” (p. 7). The process of guided reinvention enables students to discover mathematical ideas that are already known in society but that are new to them. Curricular materials should be designed so that students have an opportunity to develop an understanding of mathematical ideas through problem solving. The problems are meant to be realistic, which does not necessarily mean that they are based upon a real-world scenario but that they provide contexts that students can perceive as real. According to the theory, the problems can stem from mathematical contexts or even from the fantasy world, as long as students can use the context for guided reinvention (Freudenthal, 1971). In contrast with instruction in which students learn abstract mathematical ideas and then apply those ideas to problem solving, the RME approach situates problems at the forefront of the learning experience (Freudenthal, 1968). Van Den Heuvel-Panhuizen (2005) states,

> in RME tasks contexts are viewed in a broad sense. They may refer to everyday-life and fantasy situations in which the problems are situated, but also to the mathematical context of, for instance, a bare number problem. What is important is that the task context is suitable for mathematization—the students are able to imagine the situation or event so that they can make use of their own experiences and knowledge. (p. 3)

The activity of “mathematizing” is a key notion in RME. According to the theory, only through mathematizing are students able to learn mathematics because students are developing new mathematical knowledge through problem solving. That is, problem solving requires students to organize mathematical ideas, and in doing so, they are reinventing mathematics (Freudenthal, 1991). Gravemeijer and Doorman (1999) explain the relevance of selecting contexts so that students can apply their knowledge of the contexts and engage in mathematizing: “Well-chosen context problems offer opportunities for the students to develop informal, highly context-specific solution strategies. These informal solution procedures then, may function as foothold inventions, or as catalysts for curtailment, formalization or generalization” (p. 117). Gravemeijer and Doorman illustrate the theory with the design of a calculus course that draws upon historical perspectives and uses Galileo’s notion of velocity to introduce core concepts in calculus. The perspective offered by RME of using contexts for students to reinvent mathematics is consistent with constructivist approaches to mathematics instruction where students have opportunities to develop mathematical knowledge through problem-solving with others (Cobb, Yackel, & Wood, 1992) [2]. Gravemeijer and Terwel (2000) state, "In general, contextual problems that allow for a wide variety of solution procedures will be selected, preferably solution procedures that in themselves reflect a possible learning route" (p. 786). Lampert's (2001) case study of her experiences teaching fifth-grade mathematics exemplifies a problem-based approach for learning mathematics, where the sources of problems stem from curricular materials and, also, from issues specific to her classroom. Lampert (2001) states, "When students engage with mathematics in a problem, *the content is located in a mathematical territory where ideas are used and understood based on their relationships to one another within a field of study*" (p. 431). According to Lampert, teaching with problems requires making the connection explicit between the situation in a problem and the mathematical ideas that are used to address that situation. Teaching with problems means that teachers are actively connecting students' problem-solving approaches to the problem contexts and curricular learning goals.

The issue of selecting realistic problems that enable students to engage in guided reinvention is not trivial, and mathematics education researchers have cautioned that not all contexts are conducive to student learning. While problem contexts can increase students' access to mathematical ideas and enable students to propose various solution strategies based upon their knowledge of the context, students can also have difficulties because of their interpretation of the context. For instance, students may question solutions that are mathematically correct but do not appear to be realistic in the context, may disregard the context, or may be forced to disregard their prior knowledge of the situation because that knowledge would significantly change the mathematical solution desired in curricular materials (van Den Heuvel-Panhuizen, 2005). Students' ability to integrate their mathematical knowledge and other sources of knowledge is a sign of "mathematical power" (Kastberg, D'Ambrosio, McDermott, & Saada, 2005). However, when problem contexts are too contrived, students may not have an opportunity to integrate their knowledge. For example, Lubienski's (2000) study of the implementation of a problem-based middle school curriculum showed how students from families with low socio-economic status tended to draw upon their knowledge of the context to solve problems in ways that were not intended by the curricular developers. Boaler (2008) discusses "make-believe contexts" as a source for discouraging students from mathematics:

> One long-term effect of working on make-believe contexts is that such problems contribute to the mystery and other-worldliness of Mathland, which curtails people's interest in the subject. The other effect is that students learn to ignore contexts and

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[2] Gravemeijer and Terwel (2000) address Freudenthal's perspective on constructivism and other educational theories.

> work only with the numbers, a strategy that would not apply to any real-world or professional situation. (p. 52)

By identifying long-term effects of choosing contexts where students cannot integrate their mathematical knowledge and other sources of knowledge, Boaler establishes how curricular decisions may prevent students from finding mathematics valuable. Other studies have also identified how the selection of contexts in mathematics textbooks may perpetuate some views about mathematics. Dowling's (1998) sociological analysis of textbooks in England illustrates that the choice of contexts for mathematical problems supports what he calls "myths" about school mathematics, such as the notions that mathematics is applicable to everyday situations and that people participate in activities that require mathematical knowledge. On the other hand, the result of an exploratory study in California suggests that mathematics teachers rely on their own experiences to find real world contexts for mathematics problems since they view the contexts of textbook problems irrelevant or outdated (Gainsburg, 2008). These perspectives call into question the selection of contexts in school mathematics and lead to a critical view about how visual arts contexts in textbooks convey a sense of who is supposed to study geometry and for what purposes. The analysis of how the choice of visual arts contexts is aligned (or not) with the various discourses justifying the existence of the geometry course supports the examination of the use of contexts in the mathematics curriculum.

2. Research Questions

The four research questions guiding this study are as follows:

- What visual arts contexts are used in five mainstream geometry textbooks?
- Where are these contexts used in the textbooks (i.e., expositions, exercises, extensions, or motivations)?
- What geometry concepts are taught through a visual arts context?
- How do the visual arts contexts align with the four justifications for the geometry course?

All of these questions address whether and how geometry textbooks achieve the goals of the intended curriculum (Herbel-Eisenmann, 2007). The first two questions are descriptive and aim at identifying visual arts references and their placement in the geometry textbook. With the third research question, we attempt to list geometry concepts that have been connected to visual arts contexts, with the aim of identifying typical content areas for making these connections. With the fourth question, our goal is to see whether and how visual arts contexts are aligned with the arguments justifying the geometry course. It is possible that one textbook is more or less aligned with a particular argument. In addition, it is possible that the same visual arts context supports the achievement of various curricular goals, thus illustrating tensions in the geometry curriculum when adopting visual arts contexts. Overall, the questions use the geometry course as a basis for studying possible ways of teaching mathematics through the selection of visual arts contexts.

3. Materials and Methods

We analyzed five mainstream geometry textbooks that are aligned with the CCSSM (Table 1). The textbooks include model curricular approaches developed through funding from the National Science Foundation (Center for Mathematics Education Project, 2013), textbooks published by the major publishing companies (Dossey, Halverson, & McCrone, 2008), and a problem-based textbook written by mathematics teachers in California (Dietiker & Kassarjian, 2014).

Title	Publisher	Authors	Year
Geometry (Holt)	Holt McDougal	Larson, Boswell, Kanold, & Stiff	2012
CME Geometry Common Core (CME)	Pearson	Center for Mathematics Education Project	2013
Core connections Geometry, 2nd edition (CPM)	College Preparatory Mathematics	Dietiker, L. & Kassarjian	2014
Geometry Common Core (Pearson)	Pearson	Charles, Hall, Kennedy, Bass, Johnson, Murphy, & Wiggins	2015
Glencoe Geometry (Glencoe)	McGraw-Hill	Carter, Cuevas, Day, Malloy, & Cummins	2018

Note. The acronym in parentheses is used throughout the manuscript.

Table 1. Geometry Textbooks Analyzed

The first step of the coding process involved identifying all of the references to visual art contexts in the textbooks according to the following categories: (1) *architecture*, (2) *calligraphy*, (3) *crafts*, (4) *drawing*, (5) *film*, (6) *painting*, (7) *photography*, (8) *pottery,* (9) *sculpture,* and (10) *tiling*.[3] We identified these categories in an initial inspection of the visual arts contexts used in the textbooks, using Honour and Fleming's 1982 history of visual arts as a reference.[4] The visual arts references could appear in four different sections in a textbook: (1) *exposition,* (2) *exercises*, (3) *extensions,* and (4) *motivation*. Typically, the expositions are the narrative sections where the authors introduce the main concepts and procedures in the section. The exercises are problems with opportunities for students to apply the concepts and procedures introduced. By "extensions," we refer to additional opportunities for students to apply their knowledge of concepts and procedures that go beyond the problem sets. At times, the extensions are labelled in textbooks as "explorations" and teachers interpret these as additional curricular materials that are not required in the curriculum. Finally, a reference could be in a "motivation" section if it supports the exposition without being central to the discussion in the exposition. The motivations appear at times in the margins of a page or in the introductory section of a unit. One caveat is that the College Preparatory Mathematics (CPM) curriculum uses a problem-based approach. Consequently, there is not an exposition section as in other textbooks. Instead, the introduction of mathematical concepts and procedures appears in the exercises. Prior research on mathematics textbooks has typically focused on expositions and exercises, excluding additional resources such as the extensions and motivation sections (e.g., Otten, Gilbertson, Males, Thompson, Senk, & Johnson, 2014). We included extensions and motivations in this study to examine whether the visual arts contexts were substantially or peripherally connected to the curricular content. Finally, a reference to the visual arts context could be *inscribed* or *invoked* (Martin & White, 2005). An inscribed reference is one in which there is an explicit reference to the visual arts context. In contrast, an invoked reference is implicit, such as when a photo of a sculpture or an origami piece is included without being named as such. In addition, the reference could be to a *process* that involves visual arts (e.g., how to take a photo or make a drawing) or to a *product* (e.g., a photograph or drawing). In some contexts, there were subcategories in the coding process. These subcategories included *bridges* and *buildings* under *architecture*; *jewelry, quilting,* and *knitting* under *crafts*; and *one-point perspective* and *optical illusions* under *drawing*.

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[3] In contrast to visual arts, some problems used applied arts contexts: namely, graphic, industrial, or interior design. These references tend to be aligned with the utilitarian argument and constitute a different data set, excluded from this study.

[4] Although tiling can be a subset of architecture, references to tiling were coded separately because this is a typical topic in geometry instruction.

In the first stage of the analysis, two coders, the author and a graduate student with more than 5 years of experience teaching high school geometry who was knowledgeable about the CCSSM, independently coded the five textbooks to identify items with visual arts references. We held regular meetings to discuss the coding. At the end of the process, we reached the reliability of 42%. This value was calculated by dividing the number of agreements with the total number of references coded. We discussed our disagreements and found patterns. For example, one coder had included references to carpentry, but after discussion, we decided to exclude these references because they were not particular to visual arts. Another challenge was that some of the references were in narrative form and a coder had missed the problem when examining the textbook. References to photography were also challenging because it was unclear whether a photograph represented an example of visual art. We decided to include photographs since these showcase visual art products. Finally, we decided to include bridges as a subcategory of architecture, since architects such as Santiago Calatrava are famous for designing bridges. After these revisions, we had the reliability of 98% and reached agreement about the coding of the remaining items by consensus.

In a second layer of coding, we identified the specific content standard that the reference addressed and the domain of the CCSSM. This coding was important to answering the third research question. The two coders made an initial sorting according to CCSSM domain: (1) *Congruence*, (2) *Similarity, Right Triangles, and Trigonometry*, (3) *Circles*, (4) *Expressing Geometric Properties with Equations*, (5) *Geometric Measurement and Dimension*, and (6) *Modelling with Geometry*. Since modelling is an approach more than a specific mathematical content area, items were coded for this strand only they were identified in the textbook as addressing a learning goal pertaining to modelling. If the textbook identified the standard for the item, we took the most conservative approach and used the standard cited in the textbook in order to follow the authors' intentions. The two coders independently revised the initial sorting and identified the specific content standard addressed in a random sample of 20% of the items. We agreed on 67% of the selected items. After discussing the coding and resolving disagreements, we coded a new random sample of 20% of the items. To address previous issues with our coding, we decided that material concerning the area of 2D figures and the surface area of 3D figures pertain to the modelling domain (Standard HSG.MG. A.1), and special right triangles pertain to the similarity, right triangles, and trigonometry domain (Standard HSG.SRT. C.8). Additionally, problems that refer to applications of triangle congruence address a standard in the similarity, right triangles, and trigonometry domain.[5] We reached the

[5] Specifically, standard HSG.SRT.B.5 states, «Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.»

reliability of 81% after coding 60% of the data. After resolving disagreements, the author independently coded the remaining 40% of the items.

The next step of the analysis was to code each item according to the justification of the geometry course that is represented in the item: *formal, intuitive, mathematical,* or *utilitarian.* It is possible that an item could have more than one justification. For example, some exercises have multiple parts, and the first part could invite students to appreciate geometric patterns in an object and thus be aligned with the intuitive argument; then, in the second part, the exercise could ask students to construct a proof of geometric properties of the pattern, thus aligning the item with the mathematical argument. To account for this possibility, each item was coded as having (1) or not having (0) evidence that supports each one of the arguments. Agreements were counted per item as to whether the two coders had assigned the same code or codes to that item. The two coders coded a random sample of 20% of the items, reaching agreement on 49% of the sample. We discussed the coding and resolved disagreements before coding a new random sample of 20% of the items. We repeated the process twice, each time with a new random set of 20% of the items, reaching reliability of 67% in round 2 and 61% in round 3. One issue was that we missed statements in some exercises requiring students to "explain" or "justify" their answers, which we coded as aligned with the mathematical argument. We also decided that statements where the students were positioned as "doer" in a job-related scenario (paid or unpaid) or illustrations requiring technical knowledge (e.g., reading a floorplan) were coded as representing the utilitarian argument. We reached the reliability of 81% in our fourth round of coding a random sample of 20% of the data. I independently coded all the remaining items.

Table 2 illustrates an example of the coding sheet. In Item No. 1 there was an illustration showing a two-point perspective drawing of a house with a superimposed geometric diagram extending the sides of the house until they meet in the respective vanishing point. The diagram had labels for the vertices as it is typical in geometry diagrams. The item included three separate questions, asking students to add new line segments, write labels for the new intersection points, and draw the edges of the house that were not visible when showing the front view. The context used in this item was perspective drawing defining the terms "perspective drawing" and "vanishing point." We coded this item as aligned with the intuitive argument because it requires students to use math to appreciate a two-point perspective diagram and makes geometric knowledge accessible to students by having them identify elements in a diagram. Item No. 142, targets the same standard about the definitions of geometric terms such as "angle" and "line segment." The exercise is framed in the context of architecture by including tree diagram of two congruent trapezoidal pyramids with the labels for various angles and length measures, for students to find the measure of other congruent parts. We coded this item as aligned with the intuitive argument because it invites students' appreciation for what the authors call an archeological site." At the same time, the students are required to apply the definition of congruent figures to find the measures of lengths of segments and angles. This second goal is aligned with the mathematical argument, so this item is a case where the item assumed more than one justification for learning geometry. The reference to an archeological site suggests that mathematics is visible in the world and the work of applying to the definition of congruent figures to study the façade of the pyramids requires students to engage in mathematical reasoning. Item No. 104 provides the context of redesigning a kitchen following the guidelines of the "National Kitchen and Bath Association" by applying trigonometry. The exercise states, "Lashayia is planning to renovate her kitchen and has chosen the design at the right. Does her design conform to the National Kitchen and Bath Association's guidelines?" By positioning Lashayia in the situation of solving a problem that one may solve in a job, the item is aligned with the utilitarian argument. Item No. 120 was coded as aligned with the formal argument because it showed a photo of an outdoor theatre and students had to prove that various pairs of triangles created by the roof trusses were congruent. The architectural context provided the opportunity for students to write a proof applying triangle congruence theorems. Overall, the items in Table 2 exemplify items targeting the same standard or the same context that vary in terms of the underlying justifications for why students should study geometry represented by the four discourses.

Item No.	Textbook	Page No.	Chapter	Section No.	Section	Problem No.	Context	Strand	Standard	Justification			
										Mathematical	Formal	Utilitarian	Intuitive
1	Holt	8	1	2	Exercises	45	drawing	1	HSG.CO.A.1	0	0	0	1
90	CPM	436	7	2.4	Exercises	7.84	architecture	2	HSG.SRT.B.5	0	0	1	0
104	Glencoe	142	2	4	Exposition	1	architecture	1	HSG.CO.A.1	1	0	0	1
120	Glencoe	316	4	4	Exercises	16	architecture	2	HSG.SRT. B.5	0	1	0	0

Table 2. Example of the coding template

4. Results

The five mainstream geometry textbooks have a total of 345 items with references to visual arts contexts. Holt has the most references to visual arts with 115 items (33% of the items in all the textbooks), followed by Glencoe (79 items) and Pearson (73 items)[6]. CPM has 50 items referring to visual arts contexts. CME has the fewest references to visual arts with 28 items[7].

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[6] The Holt textbook included 37 items that repeated the same context within the same chapter; 17 of those items pertained to architecture.

7 A caveat is that the textbooks vary in the length of the text included in the exposition and motivation sections, as well as in the total number of exercises.

In answer to the first research question, Table 3 shows the number of visual arts contexts used per textbook. The context most often used in all the textbooks is architecture (43%), followed by crafts (19%), photography (14%), and drawing (11%).

Figure 1 illustrates that more than 40% of the items coded in each textbook concern architecture. More than 30% of the items in Pearson's concern drawing, but the other textbooks do not have a similar proportion of items about drawing. CPM has the most items of crafts, followed by Glencoe and Holt. In addition, Figure 1 shows that Glencoe has proportionally more items on photography than those of other textbooks. In contrast, the CPM textbook has the smallest proportion of items using photography. Most of the illustrations in CPM are drawings, unlike the illustrations in other textbooks, which include many photos. Six visual arts contexts have a proportion of less than 10% of the items in every textbook: calligraphy, film, painting, pottery, sculpture, and tiling. There was only one item for calligraphy or for pottery, showing that these two contexts are atypical in the mainstream geometry textbooks examined.

	No. of Items per Textbook (%)					
Context	*CME*	*CPM*	*Glencoe*	*Holt*	*Pearson*	Total
architecture	13 (46)	24 (48)	33 (42)	50 (43)	30 (41)	**150 (43)**
calligraphy	0 (0)	0 (0)	0 (0)	0 (0)	1 (1)	1 (0)
crafts	4 (14)	14 (28)	18 (23)	20 (17)	10 (14)	66 (19)
drawing	4 (14)	1 (2)	6 (8)	5 (4)	23 (32)	39 (11)
film	1 (4)	0 (0)	1 (1)	1 (1)	2 (3)	5 (1)
painting	2 (7)	3 (6)	3 (4)	3 (3)	2 (3)	13 (4)
photography	4 (14)	2 (4)	16 (20)	21 (18)	5 (7)	48 (14)
pottery	0 (0)	0 (0)	0 (0)	1 (1)	0 (0)	1(0)
sculpture	0 (0)	3 (6)	1 (1)	7 (6)	0 (0)	11 (3)
tiling	0 (0)	3 (6)	1 (1)	7 (6)	0 (0)	11 (3)
Total	28 (8)	50 (14)	79 (23)	**115 (33)**	73 (21)	345

Note. Some percentages do not add to 100 because of rounding

Table 3. Items with visual arts contexts per textbook

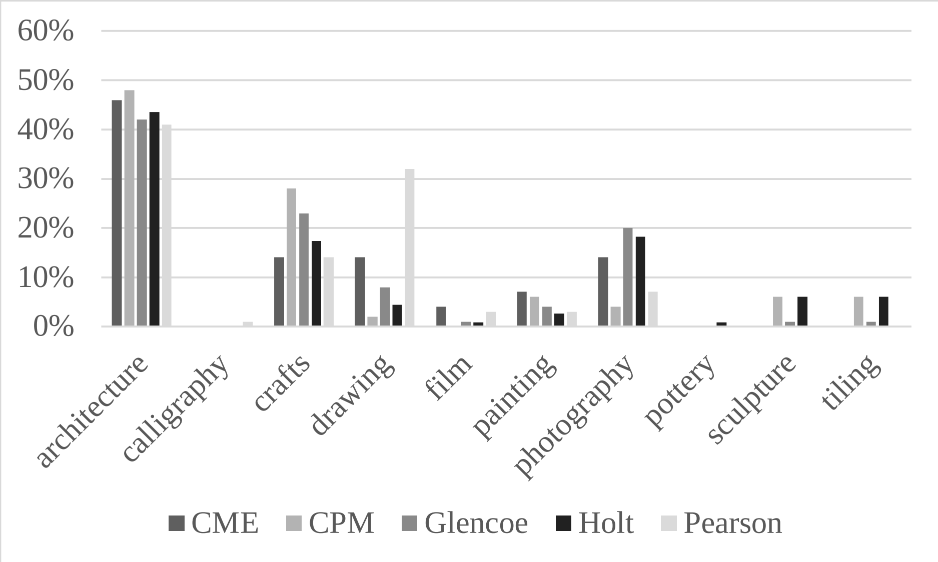

Figure 1. Percentage of items per visual arts context per textbook.

In answer to the second research question, 72% of the visual arts references appear in the exercises (see Table 4). There are 250 exercises using visual arts contexts in all the textbooks. The finding that the majority of the items were in the exercises is unsurprising since exercises typically constitute the majority of the content of a typical mathemcal textbook. Nevertheless, Pearson and Glencoe include 23% and 22%, respectively, of their visual arts references in the expositions. That is, these textbooks use visual arts contexts to introduce new geometry content. CPM, following a problem-based approach, limits the expositions to a brief paragraph, possibly explaining the low percentage of visual arts references in the expositions. CME has the highest percentage of references in motivation (36%) among all the textbooks. This result could be explained by the organization of the chapters, which include illustrations with applications of the mathematical content of the section, sometimes elaborating on or exemplifying that content. In addition, CME's introduction to similarity (chapter 4) includes 4 items referring to visual arts contexts: photography (2 items), architecture (1 item), and painting (1 item). Pearson includes 21% of its visual arts references in the extensions, being the highest among all the textbooks examined. This result is relevant because the authors included visual arts references as optional curricular topics. However, the other textbooks do not include as many visual arts references in extensions as Pearson. Pearson's high frequency of items in the explorations can be explained because 13 of the 15 items coded as extensions pertained to one exploration about one-point perspective drawings.

Section	No. of Items per Textbook (%)					Total
	CME	CPM	Glencoe	Holt	Pearson	
Exercises	**14 (50)**	**45 (90)**	**53 (67)**	**98 (85)**	**40 (55)**	**250 (72)**
Exposition	1 (4)	2 (4)	17 (22)	14 (12)	17 (23)	51 (15)
Extensions	3 (11)	2 (4)	2 (3)	0 (0)	15 (21)	22 (6)
Motivation	10 (36)	1 (2)	7 (9)	3 (3)	1 (1)	22 (6)
Total	28	50	79	115	73	345

Note. Some percentages do not add to 100 because of rounding.

Table 4. Items with visual arts contexts per textbook section

Figure 2 shows that, for all textbooks, 50% or more of the items with visual arts references are in the exercises. CPM and Holt have the highest percentage of items with visual arts references in the exercises (90% and 86%, respectively). In addition, all the textbooks include references to visual arts in their motivation.

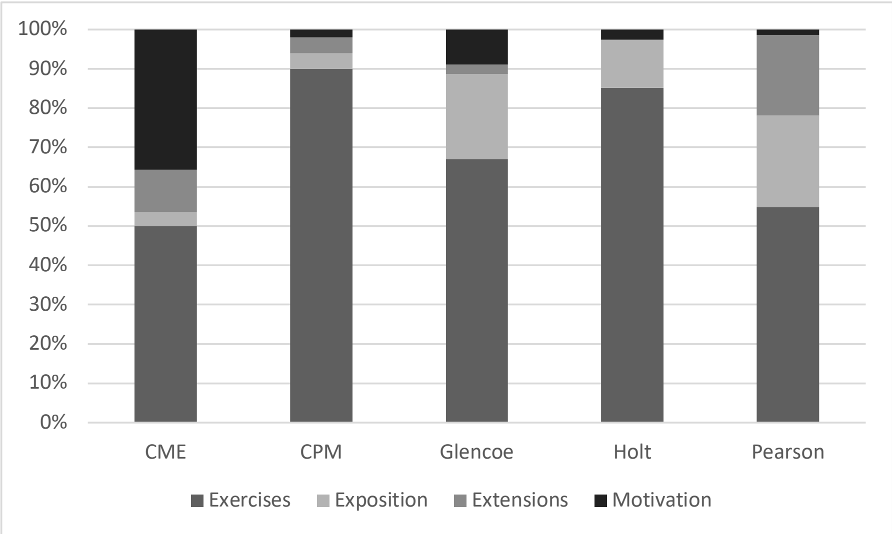

Figure 2. Percentage of items with visual arts references per textbook section per textbook.

The third research question inquires about the connections between the items with visual arts references and the mathematical content. Table 5 shows the classification of items with visual arts references by CCSSM strands. Overall, the “congruence” strand includes standards that are frequently taught through a visual arts context, with 39% of the items from all the textbooks. Two textbooks (CPM and Holt) have most of their references to visual arts contexts in the congruence strand. Across all the textbooks, the "similarity, right triangles, and trigonometry" strand is the second most targeted in the items with visual arts contexts, for a total of 34% of the items. Three textbooks have most of their items targeting this strand (CME, Glencoe, and Pearson). The standard most frequently targeted is HSG.SRT. B.5: "Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures." This standard is targeted at 47 items (14% of the total items). Notably, this standard includes topics that could also be addressed within the congruence strand but emphasizes problem solving.

Strand	No. of Items per Textbook (%)					Total
	CME	CPM	Glencoe	Holt	Pearson	
1. Congruence	4 (14)	**17 (34)**	24 (30)	**75 (65)**	15 (21)	**135 (39)**
2. Similarity, Right Triangles, & Trigonometry	**17 (61)**	14 (30)	**29 (37)**	22 (19)	**36 (49)**	119 (34)
3. Circles	0 (0)	0 (0)	13 (15)	5 (4)	4 (5)	21 (6)
4. Geometric properties with equations	0 (0)	0 (0)	0 (0)	1 (1)	0 (0)	1 (0)
5. Measurement & Dimension	2 (7)	6 (12)	10 (13)	6 (4)	17 (23)	41 (12)
6. Modelling	4 (14)	12 (24)	5 (6)	4 (3)	2 (3)	27 (8)
7. None	1 (4)	1 (2)	0 (0)	2 (2)	0 (0)	4 (1)
Total No. of Items in Standards	28	51	80	115	74	348
Total No. of Items	28	50	79	115	73	345

Note. An item could cover more than one Standard.

Table 5. Items with visual arts contexts per strand in the Common Core State Standards for Mathematics

Some textbooks emphasize the use of visual arts contexts in relation to other strands. For example, CPM has a relatively high percentage of items targeting the "modelling" strand (24%), in contrast with the other textbooks. Pearson has a relatively high percentage of items targeting the "measurement and dimension" strand (23%). It is possible that items coded as targeting other strands included a modelling approach since all the items support the examination of a context with mathematics. However, these items may not address definitions of modelling that require making assumptions and constructing a model of a situation using some parameters (Garfunkel & Montgomery, 2016). A total of 4 items did not target any specific standard, appearing in the motivation (2), exposition (1), or extensions (1). None of these items are in the exercises. This finding suggests that the visual references are mostly connected to specific geometry content to be taught by the standards.

Figure 3 shows that most of the items target strands 1 and 2. Strand 4, geometric properties with equations, is rarely represented in the items. Nevertheless, this topic is traditionally limited in the curriculum of the geometry course. Holt has the highest percentage of items in one strand (strand 1). CME and Pearson follow the second and third highest percentages of items, both in strand 2.

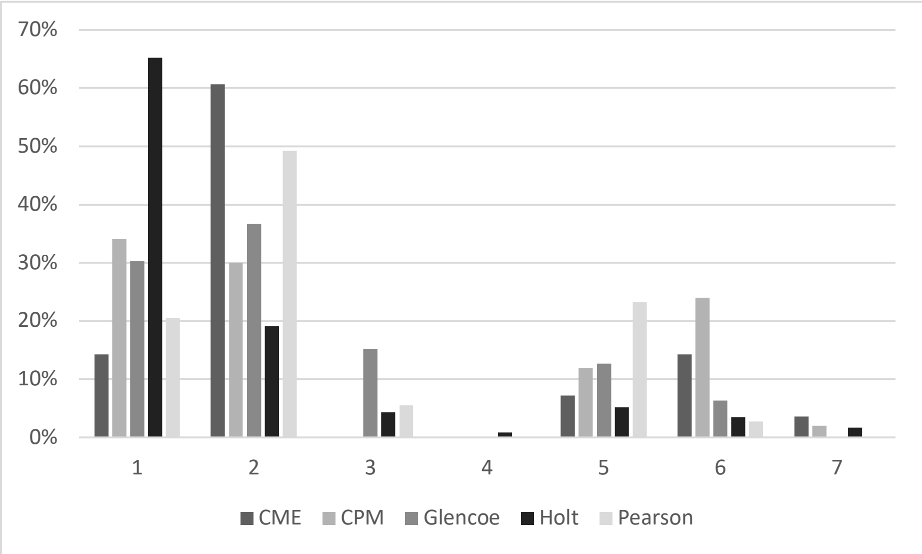

Figure 3. Percentage of items with visual arts references per strand per textbook. Strands: 1. Congruence; 2. Similarity, Right Triangles, & Trigonometry; 3. Circles; 4. Expressing Geometric Properties with Equations; 5. Geometric Measurement & Dimension; 6. Modelling with Geometry; 7. None.

In the last research question, we inquire about alignments between the four arguments supporting the geometry course and the visual arts references. Table 6 shows the results per textbook. The intuitive argument is the justification most often represented, with a total of 212 items (61% of the total items). The visual arts references in all the textbooks except CPM are more aligned with the intuitive argument than with the other arguments. Specifically, more than 40% of the items per textbook are in alignment with the intuitive argument. CPM's items are mostly aligned with the utilitarian argument (46%), followed by the intuitive argument (44%). Holt has the highest percentage of items supporting the mathematical argument (34%). The formal argument has the fewest items across all the textbooks, with no items showcasing this argument in CPM and the most items in Glencoe.

Argument	No. of Items per Textbook (%) CME	CPM	Glencoe	Holt	Pearson	Total
Formal	1 (4)	0 (0)	8 (10)	4 (3)	2 (3)	15 (4)
Intuitive	**19 (68)**	22 (44)	**45 (57)**	**82 (71)**	**44 (60)**	**212 (61)**
Mathematical	2 (7)	14 (28)	13 (16)	39 (34)	9 (12)	77 (22)
Utilitarian	8 (29)	**23 (46)**	30 (38)	23 (20)	22 (30)	106 (31)
Total Items Coded	30	59	96	148	77	410
Total No. of Items	28	50	79	115	73	345

Table 6. Items with visual arts contexts per argument (*Note*. Some items represent more than one argument).

Figure 4 illustrates that the intuitive argument was represented in many of the arguments across all the textbooks. The utilitarian argument is evident in more than 20% of the items in individual textbooks. The proportion of items aligned with the mathematical argument varies across textbooks, with Holt being the only one where it appears as the second most frequently illustrated. In contrast, the formal argument is mostly absent from the items analyzed. CPM seems to have a more balanced distribution of items among the three arguments it showcases (intuitive, utilitarian, and mathematical). CME, Holt, and Pearson emphasize the intuitive argument more than other arguments in their visual arts references.

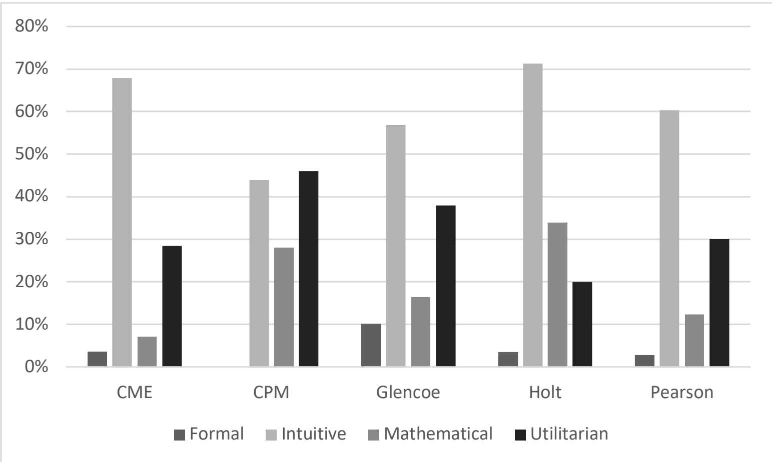

Figure 4. Percentage of items aligned with the arguments justifying the geometry course per textbook.

To illustrate the various arguments, Table 7 shows three exercises with references to the context of architecture targeting standard HSG.SRT. B.5, "Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures." Example A shows an example in which students would need to apply geometry to solve a design problem by using the guidelines established by the National Kitchen and Bath Association. This exercise models the type of problems that students would need to solve as part of the workforce, thus aligned with the utilitarian argument. Example B shows an exercise aligned with the intuitive and mathematical arguments. Example C exemplifies the formal argument since the configuration of the building's roof provides an opportunity for students to construct a proof. The notation and the terms used in this exercise are typical of this textbook when requiring a two-column proof (Herbst, 2002). The identification of geometric properties in the triangles shown in the Chrysler Building reflects the goal of teaching students to appreciate geometry in the world. At the same time, the request for students to explain their reasoning requires students to make references to geometric concepts, thus aligning this goal with the mathematical argument. While the three exercises target the same standard, they vary in the way they imply reasons for learning geometry in schools.

	Example A	Example B	Example C
Text	"The guidelines set forth by the National Kitchen & Bath Association recommends that the perimeter of the triangle connecting the refrigerator (F), stove, and sink of a kitchen be 26 feet or less. Lashayia is planning to renovate her kitchen and has chosen the design at right. Does her design conform to the National Kitchen and Bath Association's guidelines? Show how you got your answer."	"In the photo of New York City's Chrysler Building on the right, $\overline{TS} \cong \overline{ZY}$, $\overline{XY} \cong \overline{RS}$, $\overline{TR} \cong \overline{ZX}$, $\angle X \cong \angle R$, $\angle T \cong \angle Z$, $\angle Y \cong \angle S$, and $\Delta HGF \cong \Delta LKJ$. a. Which triangle, if any, is congruent to ΔYXZ? Explain your reasoning. b. Which side(s) are congruent to $\overline{JL}$? Explain your reasoning. c. Which angle(s) are congruent to $\angle G$? Explain your reasoning."	"The trusses of the roof of the outdoor theatre shown below appear to have several different pairs of congruent triangles. Assume that trusses that appear to lie on the same line actually lie on the same line. a. If $\overline{AB}$ bisects $\angle CBD$ and $\angle CAD$, prove that $\triangle ABC \cong \triangle ABD$. b. If $\triangle ABC \cong \triangle ABD$ and $\angle FCA \cong \angle EDA$ prove that $\triangle CAF \cong \triangle DAE$. c. If $\overline{HB} \cong \overline{EB}$, $\angle BHG \cong \angle BEA$, $\angle HGJ \cong \angle EAD$, $\angle JGB \cong \angle DAB$, prove that $\triangle BHG \cong \triangle BEA$."
Argument	Utilitarian	Intuitive & Mathematical	Formal
Reference	(Dietiker & Kassarjian, 2014, page 436)	(Carter et al., 2018, page 297)	(Carter et al., 2018, page 316)

Table 7. Items using architecture to address the same standard with various arguments

5. Discussion

The geometry textbooks examined use visual arts contexts to teach geometry content aligned with the Common Core State Standards for Mathematics. Most of the references to visual arts are in the exercises. The appearance of visual arts references in the exposition sections, while not prominent across the textbooks, is an important finding that shows that authors use these references to introduce the study of geometry concepts. The finding that motivation and extension sections include few references to visual arts contexts demonstrates that such references are mostly connected to the core content to be taught rather than to curricular materials that teachers may deem optional. One textbook, CME, has the highest proportion of references in the motivation sections, despite having the smallest total number of items. This finding suggests that the visual arts references in CME were mostly used as an entry point to the geometric content. Further analysis is needed to examine the cohesiveness between references in various sections. For example, if a visual arts reference in the motivation section is further developed in the exercises.

The frequent use of architecture as a context throughout the five geometry textbooks is consistent with the recommendations established by the Committee of Fifteen identifying architecture as a source for geometry problems. Architecture is also a useful context for conveying all of the arguments justifying the geometry course, with an emphasis on appreciating math in the world (the intuitive argument) and preparing students for the workforce (the utilitarian argument). Crafts and photography are the second- and third-most chosen visual arts contexts. Crafts include various types of activities such as quilting, jewelry making, and origami. The aggregation of various activities under the same code may have affected the results. The references to crafts are aligned with the intuitive argument (for students to appreciate the world around them) and at times refer to cultural practices such as origami and quilting. Crafts are also aligned with the utilitarian argument, exemplifying how the study of geometry can be useful in making crafts. Photography is another context in which the duality between the utilitarian and the intuitive arguments surfaced: students working as photographers could use geometry principles as part of their job, and students viewing photographs would better appreciate their composition because of their knowledge of geometry. In that sense, photography is a good example of visual arts being used both as a process, in which students apply geometric principles to create a photo, and also as a product, in which students learn how to read an art piece.

The prevalence of strands 1 and 2 in the visual arts references can be explained because these strands include the most standards. For example, strand 1 includes many standards about isometries (i.e., translations, rotations, and reflections), which are fundamental ideas for creating visual arts designs. In addition, problems in which students apply properties of congruence and similarity can be situated within visual arts contexts. At the same time, the limited connections with other strands deserve further attention. For example, some textbooks use jewelry making as a context for applying properties of circles, providing insights into the connections between crafts and geometry. “Measurement and dimension” is another strand where visual arts contexts can enrich the geometry curriculum. The limited connections with visual arts contexts in some strands open opportunities for investigating new connections in future geometry curriculum development.

The result that most of the items are aligned with the intuitive argument is unsurprising for various reasons. First, the proponents of the intuitive argument intended that students learn to appreciate geometry in the world. A geometry curriculum that is situated in visual arts contexts would exemplify how geometric knowledge helps students “read” the world. Second, this argument involves the use of geometry as a context for students to learn algebraic skills. The textbooks examined include exercises where students perform algebraic calculations based on examples from visual arts. Third, the proponents of the intuitive argument intended to broaden students’ access to mathematical knowledge by lessening the emphasis on proofs. The visual arts contexts were not used in alignment with the formal argument, which could reflect the limited reasoning-and-proving opportunities in U.S. geometry textbooks (Otten et al., 2014). Finally, it is possible that an emphasis on the utilitarian argument (the second-most frequent argument in the items analyzed) is insufficient for enabling students to appreciate mathematics. A school mathematics curriculum that integrates various goals can broaden students’ opportunities to appreciate mathematics beyond an emphasis on preparing for the workforce (Lyakhova, Joubert, Capraro, & Capraro, 2019). Therefore, using visual arts contexts to promote students’ connections with their experiences and to solidify their mathematical reasoning would extend students’ learning opportunities.

Conclusion

According to Stanic and Kilpatrick (1992) “The essence of curriculum is the struggle to answer the question of what we should teach” (p. 415). The selection of visual arts contexts in geometry textbooks illustrates this struggle. There are multiple reasons for students to study geometry through visual arts contexts: learning how to appreciate geometry in an origami piece, learning how to use geometric properties to design a building, applying logic to establish relationships between geometric figures in a building, or explaining the mathematical reasoning involved when studying the geometric figures in a quilt pattern. Visual arts contexts can provide an entry point for geometry students to learn mathematics. In doing so, visual arts can be a realistic context that engages students in guided reinvention. These contexts can also allow students to draw upon their knowledge bases by using their intuition about objects that they see in the world (Land, Bartell, Drake, Foote, Roth McDuffie, Turner, & Aguirre, 2018). Finding relevant contexts for students to engage in mathematics is a challenging task for teachers (González, 2017). To open up opportunities for students to engage in mathematics and to promote equitable instruction, there need to be curricular options related to students’ interests and experiences. Like the interactions between artists and mathematicians that have led to both the discovery of novel mathematical ideas and to interesting mathematical illustrations, visual arts contexts in the geometry curriculum can broaden students’ engagement and appreciation of mathematics.

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