DERYA DIANA COSAN

PRAXEOLOGICAL DIFFERENCES IN INSTITUTIONAL TRANSITION: THE CASE OF SCHOOL ALGEBRA

Abstract. The transition from lower secondary to upper secondary school is a challenging time for many students with algebra as a focal topic. In this paper, we present a new approach to this problem, based on the anthropological theory of the didactic, particularly on what we call praxeological differences between two connected institutions. The methodology involves the construction of a praxeological reference model for school algebra based on documents such as textbooks and evaluation instruments, like national exams and screening tests, from these two institutions. To illustrate this approach, the Danish transition problem in algebra between the lower and upper secondary school is examined as a case study. The results obtained by the students from these evaluation instruments are also a part of the data, to focus on knowledge actually obtained. The results from this case indicate that praxeological difference is chiefly concentrated on rules for rewriting an algebraic model.

Key words. Anthropological Theory of the Didactic, praxeological differences, praxeological reference model, arithmetic and algebra, institutional transition and transition problem

Résumé. Différences praxéologiques dans la transition institutionnelle: le cas de l'algèbre scolaire. La transition du premier au second cycle du secondaire représente un défi pour beaucoup d'élèves, l'algèbre étant un facteur principal. Dans cet article, nous proposons une nouvelle approche à l'analyse de ce problème, fondée sur la théorie anthropologique du didactique, surtout ce que nous allons appeler différences praxéologiques entre deux institutions connexes. La méthodologie implique la construction d'un modèle praxéologique de référence pour l'algèbre scolaire, fondée sur des documents provenant des deux institutions, comme les manuels et les instruments d'évaluation, comme les épreuves nationales et les tests diagnostiques. Afin d'illustrer cette approche, nous examinons le cas de la transition entre le premier et le second cycle de l'école secondaire au Danemark. Les résultats obtenus par les élèves aux évaluations font également part des données utilisées, afin d'examiner les connaissances effectives. Les résultats pour ce cas indiquent que la différence praxéologique est principalement concentrée autour des règles de traitement d'un modèle algébrique.

Mots-clés. Théorie Anthropologique du Didactique, différences praxéologiques, modèle praxéologique de référence, arithmétique et algèbre, transition institutionnelle et problèmes de transition

---

ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES, volume 29, p. 217 – 238.

1. Introduction

The transition from primary to secondary school (usually for students around the age of 12) is widely pointed out as a challenging time for students (Cantley *et al.*2021). During this time, organizational, developmental, social, and curricular difficulties or discontinuities will disrupt the students' transition and affect their subsequent learning (Cantley *et al.*, 2021).

There is not much research that maps out a specific mathematical domain or theme in the examination of the transition problem from the lower secondary to upper secondary school (Gueudet, 2016), apart from Carraher and Schliemann's (2014) research on early algebra. Gueudet (2016) emphasized that "algebra has long been the "transition topic" par excellence, marking the frontier between elementary and secondary education" (p. 18). In the Danish case, this transition appears mainly between the lower and upper secondary school, as we shall see.

Ruiz-Munzón *et al.* (2013) point out that "algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge" (p. 4). This crucial role of algebra in the acquisition and understanding of other aspects of mathematics explains the rationale behind our decision to focus on this domain.

Danish students consider the transition from the lower secondary to upper secondary school as particularly difficult in mathematics, compared with subjects like English and Danish, where many students perceive more continuity in content and difficulty (Ebbensgaard *et al.*, 2014).

This study aims to model and map the difficulties of algebra in the transition from the lower secondary to upper secondary school, with the aim to identify the specific mathematical knowledge that contributes most to the perceived differences and difficulties.

We note that in the transition from the lower secondary to upper secondary school, one may find strongly related gaps in arithmetic and algebra since elementary algebra appears at first as a more abstract point of view – or model – of certain arithmetical problems. While this extension from arithmetic to algebra begins already in lower secondary school, algebra is crucial to almost all new subjects in upper secondary school, from basic functions to calculus, analytic geometry and stochastics.

After reviewing previous research on this gap as it occurs internationally, the theoretical framework for the present study, namely the Anthropological Theory of the Didactic and praxeological differences, will be presented. We can then present the research questions of the empirical case of the paper. Subsequently, the methodology for identifying praxeological differences will be presented. Finally, the paper will analyze and shed light on the Danish case.

1.1. The transition from arithmetic to algebra

Research shows that students worldwide experience difficulties in the transition from arithmetic to algebra. For example, Filloy and Rojano (1989) point out that there is a development from arithmetic to algebraic language which relates to the notions and the forms of representation of objects and their operations. In the particular context of solving equations, Herscovics and Linchevski (1994) mention that "the inability to operate spontaneously with or on the unknown indicates the existence of a cognitive gap that can be considered a demarcation between arithmetic and algebra" (p. 63).

Arithmetic and algebra to some extent use the same symbols, but their use of these symbols is different, which can leave students feeling uncertain about their meaning (Kieran, 1990). For instance, in the earlier grades (primary school), students have an operational understanding of the equal sign, meaning they consider the equal sign as a "do something signal" (Kieran, 1981, p. 319); and as emphasized by Welder (2012), a relational understanding of the equal sign, meaning that the equal sign is used to indicate the equivalence of two expressions, is central for learning algebra. For instance, a relational understanding is necessary to manipulate and solve equations, *i.e.*, to understand that the equal sign signifies an equivalence between two expressions is crucial. The students are thus transitioning from understanding the equal sign as a connection between a calculation task and its solution, to understanding the symbol as expressing a symmetric and transitive relation (Kieran, 1990).

By considering the concept of equations, an explanation for this transition problem can appear. Students in primary school have been introduced to and worked with equations in the form $A + B = C$, which means equations where "the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations" (Filloy & Rojano, 1989, p. 19). These are referred to as the "arithmetical" notion of equality in Filloy and Rojano (1989), and methods like numerical substitutions and operating on the numerical terms only are sufficient for solving these equations.

Next, in the transition from primary to lower secondary school, students are introduced to equations like $Ax + B = Cx$, with an unknown on both sides of the equality sign. Students may no longer be able to solve equations using numerical substitutions, but solving this equation now requires operating on the entire equation (Filloy & Rojano, 1989).

In this context, Kieran (1990) points out that:

> The gap that exists between, on the one hand, problems that can be represented by equations with one unknown and that can be solved by arithmetic methods and, on the other hand, problems that are represented by equations with an unknown on each side of the equal sign and that usually must be solved by algebraic methods has been characterized by Filloy and Rojano as a didactic cut (p. 100).

It is, according to Filloy and Rojano (1989), essential to bridge this gap to enable students to transition from an arithmetical mode of functioning to an algebraic one.

Transitions have been studied from different perspectives and theories (De Vleeschouwer, 2010). This paper will examine the transition from lower secondary to upper secondary school (students of age around 16) from an institutional point of view. As De Vleeschouwer (2010, p. 155) pointed out, the transition from one institution to another is not necessarily about the existence of new mathematical content. Rather, this transition, and the problem the students experience in this context, can also be rooted in the fact that the same mathematical content is approached differently in lower secondary and upper secondary school (De Vleeschouwer, 2010). The paper will exemplify this institutional transition problem in algebra from Danish lower secondary school to Danish upper secondary school by using the Anthropological Theory of the Didactic.

2. Theoretical framework and background

The Anthropological Theory of the Didactic (hereafter ATD) introduced by Yves Chevallard (2019), aims to study human knowledge and activity, mathematical or otherwise, as phenomena that are crucially connected to the institutions that aim to develop, facilitate, and constrain them, based on the notion of praxeology (Chevallard, 2019).

According to ATD, praxeology refers to any human practice and activity and consists of two inseparable blocks, *praxis*, and a *logo* block. The praxis block (or know-how) contains one or *more types of task T*, or problems, and *techniques* τ utilized to solve these tasks (Chevallard, 2019). According to Chevallard (2019) the term 'techniques' refers to a "way of doing" tasks of type *T*. With the notation from Chevallard (2019), the praxis block is denoted as follows $\Pi = [T/\tau]$.

From an ATD point of view, no human activity can exist without any description, explanation, and justification. The required discourse on the praxis block is called logos. The logo block consists of two such discourses: a technology θ, namely the discourse utilized to describe, explain, and justify the used techniques, and a theory Θ, which refers to the formal justification of the technology (Chevallard, 2019). With the notation from Chevallard (2019), the logos block is denoted as follows: $\Lambda = [\theta/\Theta]$. The praxis block, $\Pi = [T/\tau]$, and the logo block, $\Lambda = [\theta/\Theta]$, together form a mathematical praxeological organization (also denoted mathematical organisations or mathematical praxeologies) (Barbé *et al.*, 2005) and is written in the form $\Pi \oplus \Lambda = [T/\tau] \oplus [\theta/\Theta] = [T/\tau/\theta/\Theta]$ (Chevallard, 2019).

Mathematical praxeology exhibit varying degrees of complexity: punctual, local, regional, and global ones (Bosch & Gascón, 2006). A mathematical organization (hereafter MO) is punctual if it consists of a single type of task, technique, technology, and theory. When a MO encompasses multiple punctual praxeology that shares the same technology, it is called a local MO. A regional MO comprises several local praxeology that shares the same theory. Finally, a global MO is composed of multiple regional praxeology (Barbé *et al.*, 2005).

We now consider the transition from an institution I_1 to a new institution I_2. I_1 and I_2 are two connected and neighbouring institutions, that is, students pass directly from I_1 to I_2, and depend on what they learned in I_1, at least at the entrance of I_2.

Upon entering the new institution, we assume that students are expected to arrive with a certain minimal mathematical organization MO^{I_2}. We, furthermore, let MO^{I_1} denote elements of MO^{I_2} that a certain share of the students have actually learned before leaving the institution I_1. Here, the "certain share" must be fixed and justified according to the context and aims of a given study; it could for instance be the majority of those entering I_2. We then define the praxeological difference (denoted suggestively $MO^{I_2}\backslash MO^{I_1}$ as all elements of MO^{I_2} which are not part of MO^{I_1}. Notice that these "missing prerequisites" can be entire local praxeology or just minor differences at the level of theoretical discourse, a single technique, etc. Of course, the praxeological difference could also be considered in relation to individual students and their praxeology from I_1 – a decision to include only what a majority failed to learn could reflect a pragmatic and somewhat arbitrary "average" of these individual differences. At any rate, we may often be more interested in identifying central examples than in exactness on items that are, to the expert, not expected to be central. Finally, we note that to find $MO^{I_2}\backslash MO^{I_1}$ we must determine MO^{I_2} first, and this may present greater methodological challenges (a point we return to the methodology).

We hypothesize that to describe transition problems, this concept of praxeological difference has the potential to provide a specific account of praxeological elements that contribute to causing them.

3. School algebra and ATD

Bolea *et al.* (2004) suggest that, in addition to viewing algebra as generalized arithmetic, school algebra should be interpreted as a process of algebraization of previously learned mathematical praxeology, which explains why school algebra is sometimes not treated as a distinct subject in the same way as arithmetic, geometry or statistics (Ruiz-Munzón *et al.*, 2013). Instead, it can be regarded as a general modelling tool of any school mathematical praxeology (Ruiz-Munzón *et al.*, 2013) and one may even choose to "not consider school algebra as a mathematical organization in itself, but as a way of modelling a given mathematical organization" (Bolea *et al.*, 1999, p. 137).

Ruiz-Munzón *et al.* (2013) point out that "algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge" (p. 4), which highlights the essential role of algebra as a tool to address theoretical questions that arise in various domains of school mathematics, such as arithmetic and geometry.

According to Bolea *et al.* (2004), school algebra as a modelling tool has the property of giving "answers to questions related to the scope, reliability and justification of mathematical activity which is carried out in the initial system" (p. 127) and the algebraic model holds the potential to provide a description, generalization and justification of problem-solving processes, while also gather techniques and problems that initially appear unrelated (Bolea *et al.*, 2004, p. 127).

In this paper, we introduce a relatively rough reference model of secondary school algebra which recognizes, on the one hand, that algebraic expressions often arise there as an outcome of modelling processes, but that independent work with algebraic objects is also common, for instance, in solving equations or reducing algebraic expressions that appear without a previous modelling process.

Within ATD, a praxeological reference model (hereafter PRM), is developed by considering local and regional praxeology, as well as sequences of interconnected praxeology (Bosch, 2015). Bosch (2015) notes that the explicit formulation of a PRM for subjects such as elementary algebra can serve diverse purposes. Such a model could, in particular, serve as a crucial tool for the analysis, examination, and description of the algebraic content taught and learned across diverse institutions and can furthermore be used to examine what other elements are missing or can be integrated in any teaching process (Bosch, 2015). According to Barbé *et al.* (2005), among other things, official programs and textbooks may offer "a set of mathematical elements (types of problems, techniques, notions, properties, results, etc.) that constitutes the knowledge to be taught" (p. 240-241). We can view these as elements of an MO, but the level of detail of a PRM depends on the purpose of the model, in particular the questions it is used to investigate.

For our purposes we shall only need a relatively "rough" model, which posits that school algebra at the secondary level consists of three local algebraic organizations (Hereafter AO):

1. AO_1: Set up an algebraic model, based on numerical information (That is, the tasks lead to set up an algebraic expression or equation. A simple example: if a taxi trip costs 7€ per km and there is a start fee of 9€, how can we compute the cost of an arbitrary ride?)
2. AO_2: Substituting in an algebraic model. (Here, the tasks merely involve using given models. For instance, knowing the rule $A = \pi r^2$, what is the area of a circle with radius 7?)
3. AO_3: Rewrite (operate on) an algebraic model. (For instance, knowing that $A = \pi r^2$, how can we compute the radius of a circle with a given area?)

These three algebraic praxeology together form a praxeological reference model for school algebra at the secondary level, which can be further detailed (e.g., in terms of techniques or theoretical notion I'd needed. Notice that AO_1, AO_2 and AO_3 are not independent of each other, since they share the same algebraic theoretical discourse, but they do not necessarily build upon each other.

4. Objective of this paper

Gueudet (2008) pointed out that "transition issues can be studied by focusing on mathematical organizations on different levels" (p. 246). This paper examines the transition between lower secondary and upper secondary school by studying the algebraic (praxeological) organizations and praxeological differences between these two institutions, and deals with the following questions:

How can one investigate praxeological differences between two connected institutions through the construction of a common PRM based on documents from these two institutions? In particular, what local algebraic organizations could be relevant to such differences between secondary schools?

More specifically, it has two purposes:

1. To present a general methodology for identifying praxeological differences between two neighbouring institutions based on a praxeological reference model.
2. To demonstrate this methodology in action by examining the Danish transition problem in algebra between lower secondary and upper secondary school, while using the previously introduced distinction of three local organizations in school algebra.

5. Methodology

To determine the praxeological difference at the transition between two connected institutions, I_1 and I_2, while focusing on algebra at secondary level, we can use the model introduced above. Concretely the difference can be found as the union of $AO_1^{I_2} \backslash AO_1^{I_1}$, $AO_2^{I_2} \backslash AO_2^{I_1}$ and $AO_3^{I_2} \backslash AO_3^{I_1}$. In other words, we consider the praxeology of the three main parts of school algebra separately.

At the most basic level, analyzing the algebraic praxeological difference $AO_n^{I_2} \backslash AO_n^{I_1}$ for $n = 1, 2, 3$ concretely means to identify which algebraic praxis blocks related to AO_1, AO_2 and AO_3 are expected from students in I_2, but according to data from the national exam (see later), they are not learnt in I_1 by a majority of students entering I_2, for instance because they are not assessed at the end of I_1. The analysis of what is expected by the end of I_1 is based on the exam, since the official curriculum is very vague when it comes to concrete mathematical content, and, furthermore, only has the status of "suggested goals" *(vejledende mål,* in Danish).

To find $AO_n^{I_2} \backslash AO_n^{I_1}$ for $n = 1, 2, 3$ we begin by determining $AO_n^{I_2}$ for $n = 1, 2, 3$. The general idea is to do so by analyzing documents such as textbooks and evaluation instruments (like entrance exams and screening tests) used or expected at the entrance of I_2. As mentioned in "Theoretical framework and background", the determination of $AO_n^{I_2}$ for $n = 1, 2, 3$ may present greater methodological challenges. In the Danish case, this is due to the absence of official requirements as expressed in an entrance test. It is important to highlight that the types of task found at the beginning of textbooks used for the entrance of I_2 may not necessarily reflect expected praxis blocks for students upon entering I_2. These tasks may also indicate what students are supposed to learn after becoming subject of I_2. The determination of whether solving these tasks is a new learning goal at the beginning of I_2 can be made, in part, by analyzing the level of detail in the examples presented in the textbooks. A careful examination of the specificity and thoroughness with which an example is written or explained can explicitly reveal what students are expected to already know in order to comprehend the example, as well as what new concepts are introduced therein. On the other hand, widely used screening tests at the entrance of I_2 can offer a more extensive and concrete understanding of the expectations at the entrance of I_2.

Ideally and officially, the entry level for upper secondary school corresponds to the exit level of lower secondary school, but in reality, this is not the full truth, as items appearing in review sections or screening tests demonstrate. Thus, considering tasks given to students in the first period of upper secondary school will make it possible to get closer to the actual expectations.

In the Danish context, the first two months of upper secondary school currently involve praxis and theory blocks related to linear functions and models, including algebraic and graphical representations as well as linear regression. During the period from 2017 to 2019, the Danish government required upper secondary schools to assess their students after two months from the start. These tests (a total of 8 tests from STX (The Higher General Examination Programme) called Screening test), primarily focus on linear functions and regression. Algebraic knowledge is required to solve these tasks, right from the entrance of the upper secondary school, and therefore they will be used as a main source of indications of the upper secondary school's expectations of students' algebraic knowledge at the entrance of upper secondary school. These screening tests and materials, like textbooks, are password-protected and not accessible to the public. The only publicly accessible screening test is the Silkeborg Screening Test[1].

The identification of $AO_n^{I_1}$ for $n = 1, 2, 3$ is done by analyzing the textbooks and evaluation instruments, used in I_1, and by considering the results obtained by the students from these evaluation instruments. What we look for in $AO_n^{I_1}$ for $n = 1, 2, 3$ depends on what we identified in $AO_n^{I_2}$ for $n = 1, 2, 3$. This will lead to identifying those algebraic praxeology, related to AO_1, AO_2 or AO_3, which are expected at the entrance of I_2, but they are not a part of what students actually learned in I_1. Note here that even though a type of task is present in the evaluation instruments for I_1, it is important to consider how many students actually solve this task correctly. These results will enable a more accurate indication of how many students actually master that type of task. In the Danish context, we analyzed a total of 21 exam sets posed to all students at the end of lower secondary school (9th grade), for the period 2018-2023, and by considering data from the exam results. These exam sets and exam results are password-protected and not accessible to the public. Note that $AO_n^{I_1}$ for $n = 1, 2, 3$ denote the elements of $AO_n^{I_2}$ for $n = 1, 2, 3$ that a certain share of the students has actually learned, and this "certain share" must be fixed, as mentioned in "Theoretical framework and background". In the Danish context, 70% of the students move from lower secondary school to upper secondary school, why it is clear to set "a certain share" to 70%, but it is in reality more difficult to set this fixed, as the prevalence of a type of task should also be taken into consideration, which will be illustrated later in the Danish case.

Note that in the case study, the algebraic praxeological differences will mainly be described at a technical level, as it is easier to access and takes up the most

[1] https://www.gymnasiet.dk/media/1891/screening_juni15.pdf

prominence in the written exams, while the theoretical gaps are more difficult to identify (although further studies could usefully attempt to do so).

6. A transition problem in the Danish context: Praxeological differences

6.1. Outline of a more detailed PRM

As mentioned, the praxeological reference model (PRM) for school algebra at secondary level is based on three local algebraic organizations AO_1, AO_2 and AO_3. The concrete PRM in Table 1 – based on our analysis of data as described above – is a slightly more detailed PRM for the Danish case and consists of the three local algebraic organizations, where each of them contains several types of tasks. Here a distinction is made between three praxeologies of different size and complexity. In building the PRM for the Danish case, we identify a type of task T_i for every algebraic organization and the corresponding technique τ_i used to solve T_i.

AO_1: Set up an algebraic model	AO_2: Substituting in an algebraic model	AO_3: Rewrite (operate on) an algebraic model
$T_{1,1}$: Set up a first-degree equation based on a written description with numerical data. $T_{1,2}$: Set up an algebraic model based on a geometrical situation, usually involving a diagram with symbols attached.	$T_{2,1}$: Substitution of numbers into a linear equation. $T_{2,2}$: Substitution of numbers into a given algebraic expression.	$T_{3,1}$: Rewrite (operate on) a first-degree equation. $T_{3,2}$: Rewrite (operate on) an algebraic expression

Table 1. A praxeological reference model for school algebra at secondary level in Denmark.

AO_1 consists of tasks aimed at constructing an algebraic model and AO_1 is further divided into two different types of task. AO_2 consists of tasks that can be solved by substitution in an algebraic model, both numerically and with letters and variables.

AO_3 involves tasks aimed at rewriting or operating on algebraic models, and it includes a detailed discourse and description of the techniques involved. Based on the praxeological analysis, AO_3 is divided into classes of tasks, including rewriting a first-degree equation and rewriting an expression. Both types of tasks can, for example, make use of a relatively large number of techniques related, for instance, to the commutative and distributive laws, syntactic rules governing the use or non-use of parentheses, or exponent rules. For $T_{3,1}$, certain special techniques involving operations appear in addition to these – like adding some number or expression – carried out on both sides of the equality sign. Such techniques are often used in equation solving but are not used when rewriting an algebraic expression. For that reason, we differentiate between $T_{3,1}$ and $T_{3,2}$ in the PRM. This is a main reason for the distinction of $T_{3,1}$ and $T_{3,2}$ in the PRM (Table 1).

An example of a task related to $T_{3,1}$ is:

Solve the first-degree equation: $2(x + 1) = 5x - 8$

This task can be solved by the techniques:

- τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$
- τ_2 : +, – , · or ÷ on both side of the equal sign.
- τ_3: Simplify by collecting and reducing similar terms.

An example of a task related to $T_{3,2}$ is:

Rewrite the algebraic expression: $r(5+s) + 2rs – 2r$

This task can be solved by the techniques:

- τ_1: use the distributive law $a(b + c) = a \cdot b + a \cdot c$
- τ_3: Simplify by collecting and reducing similar terms.

The following sections will outline the $AO_1^{USS} \backslash AO_1^{LSS}$, $AO_2^{USS} \backslash AO_2^{LSS}$ and $AO_3^{USS} \backslash AO_3^{LSS}$ where USS and LSS indicate Danish upper and lower secondary schools, respectively. The overall result will be that the transition problem from Danish lower secondary school to upper secondary school does not have its chief roots in $AO_1^{USS} \backslash AO_1^{LSS}$ and $AO_2^{USS} \backslash AO_2^{LSS}$ since $AO_1^{USS} \backslash AO_1^{LSS} \approx \emptyset$ and $AO_2^{USS} \backslash AO_2^{LSS} \approx \emptyset$**,** but that the transition problem is concentrated in $AO_3^{USS} \backslash AO_3^{LSS}$.

6.2. The praxeological difference: $AO_1^{USS} \backslash AO_1^{LSS}$

Tasks in $T_{1,1}$ are characterized by the students being given some situation and data, and have to assign some variables (if not given by the task formulation) and set up a model based on the given information. In AO_1^{USS}, these models are linear models, meaning they are first-degree equations or expressions. The techniques used for solving tasks in $T_{1,1}$ enable students to determine which variables are involved, to identify an initial value and a rate of change, and then setting up a linear model in the form of $y = ax + b$ with a as the rate of chance and b as the initial value.

Exercise 2 from STX 2017 (1) Screening test is a task of type $T_{1,1}$ from upper secondary school, where the students must set up a first-degree equation based on a written description. Concretely the task involves setting up a first-degree equation to describe the relationship between the temperature of the water and the time from the start of the measurements, where the initial temperature of the water was 22℃ and it increases by 7℃ per minute. As mentioned, for solving this type of task, the students have to identify the initial value and rate of change and set up a linear model.

Task of $T_{1,1}$ – and also of $T_{1,2}$ – appear every year in the final exam in Danish lower secondary school for the period 2018-2023, and by considering students' performance in the final exam at lower secondary school, we have that $T_{1,1}$ and $T_{1,2}$ are also contained in AO_1^{LSS}.

Exercise 1 from the ninth-grade exam from May 2023 is an example of $T_{1,1}$ in AO_1^{LSS}. Here, students are required to use the same technique as exercise 2 from STX 2017 (1) Screening test, as they, based on a written description, must determine which variables are involved and then set up a first-degree equation. Concretely, the student is presented with several goods whose prices have increased by 9%. The task requires the student to set up a first-degree equation that can be used to calculate the new price of a product that originally cost x DKK. 30% of the Danish ninth grade students received 2 points, and 22% received 1 point (out of 2 points) for this exercise.

Tasks related to AO_1 occur with the same prevalence in both institutions, as we have observed that the type of task related to AO_1 occurs approximately every second year in screening tests for upper secondary school and in the exam for lower secondary school. So, the prevalence of tasks related to AO_1 is the same in both institutions.

Through an analysis of material from Danish lower secondary and upper secondary school, and by considering students' performance in the final exam at lower secondary school and by considering the prevalence of tasks related to AO_1 for both institutions, it can be concluded that AO_1 occur in both institutions with the essentially same types of task and related techniques. Based on this, we claim that the praxeological difference between lower secondary and upper secondary school is not related to AO_1. In other words, $AO_1^{USS} \setminus AO_1^{LSS} \approx \emptyset$.

6.3. The praxeological differences: $AO_2^{USS} \setminus AO_2^{LSS}$

As mentioned, the praxeological reference model (PRM) for school algebra at secondary level is AO_2^{USS} involves tasks related to $T_{2,1}$ and $T_{2,2}$. These can be identified in the material from the upper secondary school, and a characteristic task is exercise 16a in Figure 1. Concretely, exercise 16a belongs to $T_{2,1}$ where the technique is to set $x = 5$ and substitute it into the function $y = 2x + 3$.

For y and x, the following relation exists: $y = 2x + 3$ What is the value of y when $x = 5$?

Figure 1. Exercise 16 (Silkeborg Screening test)

It is observed from the final exam in ninth grade in lower secondary school that tasks related to $T_{2,1}$ occur every year for the period 2018-2023. Exercise 7 from the ninth-grade exam from May 2023, which involves solving the following three equations:

- 7.1: $6x + 5 = 41$
- 7.2: $4 \cdot (x + 1) = 5x$
- 7.3: $\frac{x}{2} + 12 = 2x - 3$

This is a characteristic type of task from AO_2^{LSS}. Superficially, it appears to be of type $T_{3,1}$, but in reality – given the techniques the students use – it is not, as we shall now explain.

What characterizes tasks related to $T_{2,1}$ in AO_2^{LSS} is that they have positive coefficients and positive integer solutions from the set $\{1 \ldots 10\}$. All the equations that are identified in AO_2^{LSS} have these properties: it is sufficient to use a trial-and-error technique with the solutions in $\{1 \ldots 10\}$ and thus get the solution with techniques for $T_{2,1}$, without algebraic operations. The tasks, 7.1, 7.2 and 7.3, are solved correctly by, respectively, 80%, 47% and 29% of the Danish students in the final exam at lower secondary school. Based on these observations, we claim that Danish lower secondary school students use a trial-and-error technique with the solutions in $\{1 \ldots 10\}$ for solving the tasks 7.1, 7.2 and 7.3. We claim that it is more difficult for the students to use substitution with the solutions in $\{1 \ldots 10\}$ in tasks 7.2 and 7.3, since parentheses and fractions are involved, which could be more difficult to calculate, which is why fewer students can solve tasks 7.2 and 7.3 correctly. Because if the students had used techniques such as the commutative and distributive laws, syntactic rules governing the use or non-use of parentheses, or exponent rules, the tasks, 7.1, 7.2 and 7.3, would be equally "easy" to solve, since they are all first-degree equations and thus have more or less the same correctness among the students.

Note also that substitution with solutions in $\{1 \ldots 10\}$ is a predominant technique in lower secondary school, even in tasks that on the surface looks like tasks related to $T_{3,1}$ (e.g. the tasks 7.1, 7.2 and 7.3). Tasks such as tasks 7.1, 7.2 and 7.3 occur every year in the final exam in lower secondary school with the same progression, *i.e.*, where the first task always has a higher correctness among the students and where questions 2 and 3 always involve fractions and parentheses.

Exercise 15.2 from the ninth-grade exam from May 2023 is an example of a task belonging to $T_{3,1}$ in AO_3^{LSS} and it was solved correctly by 35% of the Danish ninth grade students. The task involves determining the area of the base in a pyramid with a rectangular base, given its volume, 40 cm^3, and height, 12 cm. In the task, a sketch of the pyramid is given with a rectangular base, where the base dimensions are 2 cm and 4 cm, and the height from the base to the apex of the pyramid is 9 cm. To find the area of the base, the students have been given the formula

$V = \frac{1}{3} \cdot h \cdot G$ where *V* is the volume of a pyramid, *h* is the height of the pyramid and *G* is the area of the pyramid's base. On the surface, the task gives the impression that students need to rewrite the expression and isolating *G*, but what is characteristic of such tasks in ninth-grade exams is that they all have an integer solution, which is why rewriting does not become a prevailing technique among students, according to the guidance offered to the teachers and the exam results.

Through an analysis of material from Danish lower secondary and upper secondary school, and by considering students' performance in the final exam at lower secondary school, it can be concluded that the same types of task and techniques related to AO_2 occur at Danish lower secondary and upper secondary school. We can therefore conclude that the praxeological difference between lower secondary and upper secondary school is not related to AO_2. Therefore, we conclude that $AO_2^{USS} \backslash AO_2^{LSS} \approx \emptyset$.

6.4. The praxeological differences: $AO_3^{USS} \backslash AO_3^{LSS}$

AO_3^{USS} involves tasks related to $T_{3,1}$ and $T_{3,2}$. These can be found in the material from the upper secondary school, and a characteristic task is exercise 6 from STX 2017 (1) Screening test.

The exercise is about students being presented in an attempt to solve the equation $3x + 2(x + 1) + 7 = 5$ based on the following series of rewrites:

$$3x + 2(x + 1) + 7 = 5$$

$$3x + 2x + 1 + 7 = 5$$

$$5x + 8 = 5$$

$$5x = 3$$

$$x = \frac{5}{3}$$

and the students are tasked with identifying and describing the mistakes made in these rewrites. Concretely, this exercise belongs to $T_{3,1}$ and tasks related to $T_{3,1}$ in AO_3^{USS} have in common that solving them require the use of techniques where an operation on or with the entire equation is needed.

Notice that the classification of tasks related to either AO_2 or AO_3 is determined by observing what students actually do when they solve an equation. If an equation is solved by using a trial-and-error technique with the solutions in {1…10} and thus without algebraic operations, it can be characterized as a task in AO_2. However, if techniques involving operation in or with the entire equation are done, then the task can be classified as a task in AO_3. For example, the tasks 7.1, 7.2 and 7.3 from the ninth-grade exam from May 2023 can be classified as either AO_2 or AO_3, but we classify it as a part of AO_2 since it has solutions in {1…10}. For exercise 6 from STX 2017 (1) Screening test, the situation is different.

This exercise illustrates a prevalent type of task, related to $T_{3,1}$, that upper secondary school students are expected to be able to solve at the entrance of upper secondary school.

This task can be solved by the techniques:

- τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$
- τ_2 : +, - , · or ÷ on both side of the equal sign.
- τ_3: Simplify by collecting and reducing similar terms.

From an analysis of textbooks used at the entrance of the upper secondary school, tasks related to $T_{3,1}$ in AO_3^{USS}, are identified as tasks that students should be able to solve at the beginning of upper secondary school.

For example, in an exercise from MAT STX textbook introductory phase, students are tasked with solving the following three equations by hand:

1. $3(14 + x) = 9$
2. $-3 \cdot x = 5$
3. $7 - 2x = 3x - 3$

While these tasks might initially seem like previous ones *i.e.*, tasks 7.1, 7.2 and 7.3 from the ninth-grade exam from May 2023 from lower secondary school, there are notable differences. Students move from lower secondary school, where a trial-and-error technique suffices for solving equations with positive coefficients and positive integer solutions, to upper secondary school, where the techniques (τ_1 and τ_2) to manipulate and operate algebraically become necessary to solve first-degree equations; moreover they can have both negative coefficients, negative integer solutions, and non-integer solutions (as the equations of MAT STX textbook).

Figure 2 shows some tasks, used in the entrance of the upper secondary school, which are related to $T_{3,2}$ in AO_3^{USS}.

Simplify the following expressions as much as possible:

1. $\frac{a^4 \cdot b^3}{a^2 \cdot b}$
2. $(a-2b)^2$
3. $(x-1)(x+2)$

Figure 2. Exercise 1, 2, and 3 (Silkeborg Screening test)

Exercise 1 in Figure 2 can be solved by the techniques related multiplication of fractions and exponent rules such as τ_4: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ and τ_5: use the quotient rule $\frac{a^m}{a^n} = a^{m-n}$, while exercise 2 can be solved by the technique of squaring a binomial τ_6 : $(a-b)^2 = a^2 + b^2 - 2ab$. Finally, exercise 3 can be solved by the technique τ_1: use the distributive law $a(b + c) = a \cdot b + a \cdot c$.

So AO_3^{USS} consists of types of tasks related to $T_{3,1}$ and $T_{3,2}$ with corresponding techniques τ_1, τ_2, τ_3, τ_4, τ_5 and τ_6.

Very few types of tasks related to $T_{3,1}$ and $T_{3,2}$ exist in AO_3^{LSS}. We have observed that tasks related to $T_{3,2}$ in AO_3^{LSS} involve tasks where students are not required to perform a rewriting of an algebraic expression themselves, but instead, they need to explain a rewriting of an algebraic expression. Notice that out of 10 final exams with aids (where each exam consists of an average of 20 tasks) for the period 2018-2023, this type of task related to $T_{3,2}$ has occurred in 5 out of 10 final exams as one out of the 20 tasks. Therefore, this type of task occurs to a lesser extent in the final exam for lower secondary school. An example of this type of task is exercise 6.3 from ninth-grade exams from May 2021. The exercise is about students being presented in an attempt to rewrite the expression $n^2 - (n+1) \cdot (n-1)$ based on the following series of rewrite:

$$\begin{aligned} n^2 - (n+1) \cdot (n-1) &= n^2 - (n^2 - n + n + 1) \\ &= n^2 - n^2 - n + n + 1 \\ &= 1 \end{aligned}$$

and the students are tasked with explaining the mistakes made in these rewrites.

By considering students' performance in the final exam at lower secondary school, we shall now examine the extent to which these tasks were solved correctly by students, which is essential to consider in the analysis of matter learnt.

To solve exercise 6.3 from ninth-grade exams from May 2021, where the students' aim is to explain the mistakes that are made in an algebraic rewriting, students need to have acquired the technique τ_1: use the distributive law $a(b + c) = a \cdot b + a \cdot c$. 5% of the students received 3 points, and 15% received 2 points (out of 3 points) for exercise 6.3, which could indicate that although a few tasks of type $T_{3,2}$ exists in AO_3^{LSS}, they can actually only be solved by very few students.

Exercise 6.3, which is a task related to $T_{3,2}$ in AO_3^{LSS}, is correctly solved by a maximum of 20% of the students. This type of low correctness, with a maximum of 35% in general, in the final exam among the Danish lower secondary students is a result that can also be observed in other tasks related to AO_3^{LSS}. It is therefore possible, based on the low student performance in the few and very unambitious exam tasks, to conclude that it is only a small minority that acquires parts of AO_3^{LSS} in lower secondary school.

As mentioned in the previous section, it is possible to observe tasks related to solving a first-degree equation in the final exam for Danish lower secondary school. However, since these equations have a solution in $\{1 \dots 10\}$, we chose to categorize these as tasks belonging to $T_{2,1}$ in AO_3^{LSS}. This gives that tasks which at first sight can be characterized as tasks related to $T_{3,1}$ in AO_3^{LSS}, do not belong to it, which is why $T_{3,1}$ is almost not to be found in AO_3^{LSS}.

In conclusion, AO_3^{USS} consists of tasks related to $T_{3,1}$ and $T_{3,2}$. $T_{3,1}$ contains types of tasks related to solving a first-degree equation (with negative coefficients, negative integer solutions, and real solutions) by operating on or with the entire equation, while $T_{3,2}$ contains types of tasks related to rewriting and operating on an algebraic expression, which is not limited to linear expressions. On the surface, by observing official tests such as the final exam for ninth grade, we see that in lower secondary school, there are tasks related to solving and operating on first-degree equations, and to rewrite expressions. However, the reality in lower secondary school is that all tasks related to solving first-degree equations can be solved by using a trial-and-error method with the solutions in $\{1 \dots 10\}$ and thus without algebraic operations. So, in lower secondary school, students can achieve full points by solving a first-degree equation without operating on the equation at all and the problem of lower secondary school is also that tasks related to AO_3 are solved by a few students. As we observed, AO_3^{USS} involves numerous rules and techniques, whereas AO_3^{LSS} is almost empty. When examining the very few types of tasks related to $T_{3,2}$ in AO_3^{LSS}, we noticed that they do not involve students working with expressions, as is the case with $T_{3,2}$ in AO_3^{USS}. Instead, students are only required to explain the simplification of expression rather than performing the simplification using techniques from $T_{3,2}$. So, based on these considerations, we can conclude that the transition problem from Danish lower secondary to upper secondary school is concentrated in to $AO_3^{USS} \backslash AO_3^{LSS}$.

Concretely, we can now say that the transition problem between Danish lower secondary school and upper secondary school lies in the fact that AO_3^{LSS} is almost empty while AO_3^{USS} contains many types of tasks and corresponding techniques. This means that $AO_3^{USS} \backslash AO_3^{LSS}$ is where the praxeological difference is greatest compared with $AO_1^{USS} \backslash AO_1^{LSS}$ and $AO_2^{USS} \backslash AO_2^{LSS}$. This is thus the reason for the significant algebraic gap and thus the transition problem between these two institutions.

7. Discussion

The present study has aimed to examine transition problems in algebra across institutions. To address the transition problem, our main point in this paper was to present a new theoretical notion *praxeological differences* within ATD, as a promising way to understand and describe a transition problem between two neighbouring institutions. Furthermore, we have presented a general methodology for identifying praxeological differences in algebra between neighbouring institutions, using a praxeological reference model for school algebra. Praxeological differences and the corresponding method can be useful in other institutional transitions as well, such as the transition from primary to lower secondary school, and for other mathematical domains with their respective praxeological reference model. A methodological challenge is that it can be very difficult to identify MO^{I_2}, as there is not always concrete material or tests used at the entrance to I_2. In the present study, this was observed in the Danish case. Another challenge is that it is difficult to assess the knowledge acquired by the lower secondary school students without access to their exam results. The term praxeological difference is a useful concept for use on an individual level, but when considering transition problems, it is the sum of all individuals' actually learned knowledge that is central, which is why access to data such as exam results can be important.

A methodical choice we have made in determining the praxeological difference between lower secondary school and upper secondary school, in a Danish context, is to focus on the praxis block. There are two reasons for this. Firstly, we observe that the praxis block, at the technical level, is what creates the biggest challenges for the students. Furthermore, the praxis block has a greater presence in the materials of both institutions, and it is difficult to identify the logos block.

For the Danish case, we have observed that the first-degree equation exists in the material from lower secondary school, but even though they are all characterized by having solutions in $\{1\ldots10\}$ and can be solved by a substitution, we observe that there is also a significant variation in how many students solve the tasks correctly. Exercise 7 from the ninth-grade exam from May 2023 is a task with three different first-degree equations of increasing complexity. The tasks, 7.1, 7.2 and 7.3, are solved correctly by, respectively, 80%, 47% and 29% of the Danish students in the final exam at lower secondary school. The decrease in the number of students who have solved the task correctly may, according to Filloy and Rojano (1989), be because students are used to working with equations in the form $Ax + B = C$, where numerical substitution is sufficient to solve this type of equation. However, Task 7.2 and 7.3 from the ninth-grade exam from May 2023 are of the form $Ax + B = Cx$, and according to Filloy and Rojano (1989), students can no longer use numerical substitution for this type of equation. But this is not what we observe in the Danish case. Even equations of the form $Ax + B = Cx$ in Danish lower secondary school have solutions in $\{1\ldots10\}$, so these equations are also solved with a trial-and-error technique. So Danish students solve complicated equations, as termed by Filloy and Rojano (1989), with a trial-and-error technique and substitution, and if they calculate incorrectly during this substitution, they can end up solving the equation incorrectly. Therefore, we claim that Danish lower secondary students do not solve first-degree equations incorrectly because the equations become more complicated, as Filloy and Rojano (1989) point out, since the technique remains the same; however, students may calculate incorrectly, for example, within parentheses or with fractions when using a trial-and-error technique with solutions in $\{1\ldots10\}$.

Based on the concept of praxeological differences and praxeological reference model, we can state that the transition problems in school algebra from Danish lower secondary school to upper secondary school is due to praxeological difference $AO_3^{USS}\backslash AO_3^{LSS}$. According to Kieran (1990), this may be because the transition from an operational understanding to a relational understanding of the equal sign has not succeeded, as mastery of AO_3 requires a relational understanding. As indicated by Filloy and Rojano (1989), we can assert that Danish students complete primary school with an arithmetical notion of equality, which could be the reason why the praxeological difference $AO_3^{USS}\backslash AO_3^{LSS}$ arises.

There are so many techniques in AO_3 that it is probably the most important, compared to AO_1 and AO_2, which contain fewer techniques. We have observed that there are few tasks related to AO_1 and AO_2 in both institutions, and these tasks were solved correctly by a limited number of students in lower secondary school. Consequently, AO_1 and AO_2 do not occupy much space in both institutions. We, therefore, found that the greatest praxeological difference, and where we believe the transition problem lies, is at $AO_3^{USS}\backslash AO_3^{LSS}$.

Transitional problems are therefore not directly caused by the tasks that the fewest students solve correctly in an institution. It is equally about the prevalence of a certain type of task. AO_3 is highly dominant and prominent in upper secondary schools but almost entirely absent in lower secondary school. Consequently, the praxeological difference $AO_3^{USS} \backslash AO_3^{LSS}$ is the largest and, thus, the most important compared to $AO_1^{USS} \backslash AO_1^{LSS}$ and $AO_2^{USS} \backslash AO_2^{LSS}$. Therefore, if the prevalence of a certain type of task is high in I_2 and almost absent in I_1, the praxeological difference $MO^{I_2} \backslash MO^{I_1}$ will be large.

8. Conclusion

The present study contributes to the Anthropological Theory of the Didactic by introducing the concept of *praxeological differences* between two neighbouring institutions and presenting a general methodology for identifying these differences based on a praxeological reference model. We assert that praxeological differences, denoted as $MO^{I_2} \backslash MO^{I_1}$, and the corresponding methodology has the potential to address the transition problem between two connected institutions, denoted as I_1 and I_2. We have argued that the praxeological reference model for algebra consists of three local algebraic praxeology; AO_1: Set up an algebraic model, AO_2: Substituting in an algebraic model and AO_3: Rewrite (operate on) an algebraic model.

Applying this general methodology and the praxeological reference model for algebra, we examine the Danish transition problem in algebra from lower secondary school to upper secondary school by identifying praxeological differences: $AO_1^{USS} \backslash AO_1^{LSS}$, $AO_2^{USS} \backslash AO_2^{LSS}$ and $AO_3^{USS} \backslash AO_3^{LSS}$. Our findings indicate that the transition problem is primarily attributed to the praxeological difference $AO_3^{USS} \backslash AO_3^{LSS}$.

Acknowledgments

The author would like to thank her supervisor, Professor Carl Winsløw, University of Copenhagen, for his valuable suggestions and instructions in this study.

Bibliography

BARBÉ, J., BOSCH, M., ESPINOZA, L. & GASCON, J. (2005). Didactic restrictions on teachers practice - the case of limits of functions in Spanish high schools. *Educational Studies in Mathematics, 59*, 235–268. https://doi.org/10.1007/s10649-005-5889-z

BOLEA, P., BOSCH, M. & GASCÓN J. (2004). Why is modelling not included in the teaching of algebra at secondary school? In Mariotti, M. A. (Ed.) (2004). *European Research in Mathematics Education III: Proceedings of the Third Conference of the European Society for Research in Mathematics Education* (p. 125–133). University of Pisa and ERME.

BOLEA, P., BOSCH, M., & GASCÓN, J. (1999). The role of algebraization in the study of a mathematical organization. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. II, p. 135–145). Forschungsinstitut fuer Mathematik didaktik.

BOSCH, M. (2015). Doing Research Within the Anthropological Theory of the Didactic: The Case of School Algebra. In S. Cho (Eds), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (p. 51–69). Springer. https://doi.org/10.1007/978-3-319-17187-6_4

BOSCH, M. & GASCÓN, J. (2006). Twenty-Five Years of the Didactic Transposition. *In Bulletin of the International Commission on Mathematical Instruction*, *58*, 51–65.

CANTLEY, I., O'MEARA, N., PRENDERGAST, M., HARBISON, L., & O'HARA, C. (2021). Framework for analysing continuity in students' learning experiences during primary to secondary transition in mathematics. *Irish Educational Studies*, *40*(1), 37–49. https://doi.org/10.1080/03323315.2020.1779108

CARRAHER, D., & SCHLIEMANN, A. D. (2014). Early algebra teaching and learning. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (p. 193–196). Springer. https://doi.org/10.1007/978-94-007-4978-8_54

CHEVALLARD, Y. (2019). Introducing the Anthropological Theory of the Didactic: An attempt at a Principled Approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114. https://doi.org/10.24529/hjme.1205

DE VLEESCHOUWER, M. (2010). An institutional point of view of the secondary–university transition: The case of duality. *International Journal of Mathematical Education in Science and Technology*, *41*, 155–171. https://doi.org/10.1080/00207390903372445

EBBENSGAARD, A. H. B., JACOBSEN, J. C., & ULRIKSEN, L. (2014). *Overgangsproblemer mellem grundskole og gymnasium i fagene dansk, matematik og engelsk [Transition problems between lower secondary school and upper secondary school in the subjects of Danish, Mathematics, and English]. IND's Skriftserie, Nr. 37*. University of Copenhagen. http://www.ind.ku.dk/publikationer/inds_skriftserie/2014-37

FILLOY, E., & ROJANO, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

GUEUDET, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, *67*, 237–254. https://doi.org/10.1007/s10649-007-9100-6

GUEUDET, G. (2016). Survey on the State of the Art. In G. Gueudet, M. Bosch, A. A. diSessa, O. N. Kwon & L. Verschaffel (Eds.), *Transitions in Mathematics*

Education (p. 1-34). Springer International Publishing. https://doi.org/10.1007/978-3-319-31622-2_1

HERSCOVICS, N., & LINCHEVSKI, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27,* 59–78.

KIERAN, C. (1981). Concepts associated with the equality symbol. *Educational studies in Mathematics*, *12*(3), 317–326.

KIERAN, C. (1990). Cognitive Processes Involved in Learning School Algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education* (p. 96–112). Cambridge University Press. https://doi.org/10.1017/CBO9781139013499.007

RUIZ-MUNZÓN, N., BOSCH, M., & GASCÓN, J. (2013). Comparing approaches through a reference epistemological model: the case of school algebra. In B. Ubuz, C. Haser & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (p. 2870–2879). Middle East Technical University and ERME.

WELDER, R.M. (2012). Improving Algebra Preparation: Implications From Research on Student Misconceptions and Difficulties. *School Science and Mathematics*, *112*, 255–264. https://doi.org/10.1111/j.1949-8594.2012.00136.x

DERYA DIANA COSAN

University of Copenhagen

ddc@ind.ku.dk