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ACCUMULATIVE THINKING AS AN INTUITIVE BASE FOR THE CONCEPT OF INTEGRAL

Abstract. Integral calculus presents persistent challenges for students in general and for those transitioning from secondary to tertiary education in particular. This study examines the development of "Accumulative Thinking" as a foundation for understanding integration. In a purposely designed learning activity, pairs of Grade 11 students explored accumulation concepts using the context of water flowing into a pool. Using the Abstraction in Context framework, we analysed students' processes of constructing knowledge during this activity. Our findings indicate that most students constructed elements of Accumulative Thinking, preparing them for future studies; the findings also demonstrate how real-world contexts can facilitate the development of Accumulative Thinking.

Keywords. Calculus, integration, accumulation, accumulative thinking, construction of knowledge, secondary-tertiary transition.

Résumé. Le calcul intégral présente des défis persistants pour les étudiants en général et pour ceux qui passent de l'enseignement secondaire à l'enseignement tertiaire en particulier. Cette étude examine le développement de « pensée accumulative » comme fondement à la compréhension de l'intégration. Au cours d'une activité d'apprentissage spécialement conçue, des paires d'élèves de la 11ème année scolaire ont exploré des concepts d'accumulation en utilisant le contexte d'eau coulant dans une piscine. En utilisant le cadre d'« abstraction en contexte », nous avons analysé les processus de construction de connaissances des élèves au cours de cette activité. Nos résultats indiquent que la plupart des élèves ont construit des éléments de pensée accumulative, les préparant ainsi à des études futures ; les résultats montrent également comment un contexte réel peut faciliter le développement de la pensée accumulative.

Mots-clés. Calcul différentiel et intégral, accumulation, pensée accumulative, construction de connaissances, transition secondaire-tertiaire.

Introduction

Learning and teaching calculus in general, and integral calculus in particular, is a challenging issue at secondary and tertiary levels. This paper deals with a preparatory learning activity offering students an opportunity to develop ways of thinking that will be useful when studying integration with an accumulation approach. We call such thinking *Accumulative Thinking*. There are at least two ways to consider the didactical base for learning accumulation, as 'adding up pieces' and as derived from Rate of Change (Ely & Jones, 2023). In the feasibility study presented here we

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mainly adopted the second way. The design of the learning activity and the investigation of its implementation were informed by the theoretical framework of Abstraction in Context (Dreyfus et al., 2015). The research reported in this paper was conducted in continuous discussion and cooperation between an experienced high school teacher of mathematics (the first author), an experienced tertiary teacher of mathematics (the second author), and researchers (the second and third authors). The findings of the research can serve as a didactic and methodological approach to teaching and learning integral calculus throughout the educational continuum to define a teaching perspective at the secondary level, to facilitate the transition to the tertiary level, and to serve as a meaningful base at the tertiary level.

1. Background

1.1. The integral as an accumulating quantity

The concept of integral $\int_a^b f(x)dx$ can be approached via the notion of *anti-derivative* and/or via the notion of *accumulation*. The *anti-derivative* approach uses the difference between the anti-derivative values at the upper boundary b and the lower boundary a of the integral. The Function $F(x)$ is an anti-derivative of the function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of $f(x)$. If $f(x)$ is a continuous function in $[a, b]$, there exist antiderivatives of f and if $F(x)$ is one of the anti-derivatives of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

The *accumulation approach* is based on quantitative reasoning (Thompson, 2022); it considers the function $f(x)$ to be the rate of change (RoC) of a quantity Q with respect to the variable of integration x, and as a consequence, $\int_a^b f(x)dx$ represents the amount of Q accumulated as x varies from a to b. This connection is established by the approximation of the integral as a sum of products according to Riemann, as in

$$\int_a^b f(x)dx \cong \sum_{k=1}^{n} f(a + (k-1)\Delta x) \cdot \Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

Indeed, for given n, since $f(a + (k-1)\Delta x)$ is the rate of change of Q at the beginning of the k^{th} interval of length Δx, the product $f(a + (k-1)\Delta x) \cdot \Delta x$ is (an approximation of) the bit of quantity accumulated in the k^{th} interval, and the sum of these bits over all intervals is the total quantity accumulated from a to b.

An accumulation function approximating the total quantity accumulated from a to x for any $a \leq x \leq b$, is then defined naturally by

$$A(x) = \sum_{k=1}^{m} f(a + (k-1)\Delta x) \cdot \Delta x + f(a + m\,\Delta x)(x - (a + m\Delta x))$$

where m is chosen so that $m\Delta x \leq x < (m+1)\Delta x$ (Thompson & Ashbrook, 2019). The function $A(x)$ interpolates linearly between the values of the above Riemann sum at the endpoints of the intervals $x = a + k\Delta x$.

1.2. Research on integration as accumulation

In the special case where x is time, the products are of the form 'length of time interval' × 'RoC with respect to time in that interval' (for example speed), and it is particularly easy to grasp the quantitative nature of the bits that accumulate (for example, bits of displacement). According to Thompson and Silverman (2008), in order for students to see that the area under the graph of f represents a quantity, it is important to understand that the accumulated quantity consists of incremental bits which are created multiplicatively by two quantities the form $f(x) \cdot \Delta x$.

Researchers have reported difficulties of students at the secondary as well as at the tertiary level with the integral concept (e.g., Bressoud, 2009; Ely & Jones, 2023; Jones, 2015a; Orton, 1983; Rösken & Rolka, 2007). Such difficulties arise, for example, when dealing with the integral $\int_a^b f(x)dx$ when $f(x)$ is negative or when b is less than a (Orton, 1983). When finding the area between a given graph and the horizontal axis, where the graph is partially above and partially below the axis, it was found that the students who solved correctly could not explain why. Kouropatov and Dreyfus (2013) report that only 9% of 12$^{\text{th}}$ grade high school students in their study agreed with the claim that *if a continuous function $f(x)$ is negative, then its definite integral is negative*; 58% answered that the integral will be positive and explained that an integral is an area, and an area has a positive value. Even high-ability students rarely acquire comprehension regarding the central concepts of calculus; instead, in the best case, formal techniques allow them to answer exercises (e.g., Thompson & Harel, 2021).

According to Thompson and Silverman (2008), in order to hold a well-structured understanding of accumulation function, students need to coordinate three variables simultaneously: the changing value of x, the value of the integrand f that changes accordingly, and the value of the quantity accumulated up to x, which derives from these changes of x and f, and which changes according to both. Hence, one of the difficulties in understanding accumulation function arises from understanding accumulation as a function which depends on another function. Students typically encounter such complex covariation for the first time when learning about accumulation.

Constructing the integral concept based on the idea of accumulation has been shown to be beneficial (Carlson, Smith & Persson, 2003; Kouropatov, 2016). Kouropatov (2016) built and researched a teaching unit for the integral concept based on approximation and accumulation; he concluded that this approach has clarified the connection between integration and differentiation for the students and has allowed them to realize the mathematical meaning of the integral. Carlson et al. (2003) developed notions of accumulating quantities, accumulation functions and the Fundamental Theorem of Calculus (FTC). Their students examined in detail the incremental accumulation of various quantities, tying the idea of accumulation to the notation. This approach resulted in a high success rate in terms of students' conceptions of accumulation functions and their understanding of the FTC.

The understanding of integration as an accumulation process lies at the heart of the comprehension of the underlying mathematical ideas as well as many applications (Ely & Jones, 2023). The accumulation function has the potential to serve as a model for different situations that describe continuous processes of everyday life. Conversely, many such processes have the potential to illustrate accumulation.

In summary, research has established that the approach to integration based on the idea of accumulation is efficient; in this paper, we propose to prepare such an approach at the secondary level, thus contributing to a smooth transition to the tertiary level. We will argue and present evidence that this can be done by incorporating intuitive and tangible real-world examples and emphasizing the strong connection between accumulation and RoC.

1.3. Extra-mathematical context

According to Gravemeijer and Doorman (1999), context problems are defined as problems whose problem situation is experientially real to the student. In their research they discuss the role of context problems in calculus courses as they are used in the Dutch approach (known as Realistic Mathematics Education).

In their article, the extra-mathematical context of velocity and distance from a given starting point is used to approximate the accumulated amount (distance from the starting point) from a given function representing the rate of change (speed as a function of time). The approximation is made using a step function by assuming a constant velocity for each interval, which equals the value of the velocity function at the left (initial) endpoint of the interval. This extra-mathematical context offers students a way to act and reason meaningfully, and hence to develop informal, context-dependent strategies; these informal strategies may help them later to develop generalization and formalization of accumulation (Gravemeijer & Doorman, 1999).

The effect of extra-mathematical context on problem-solving has been widely studied in various subjects, for example computer programming (Leinonen et al., 2021), algebra (Bottge, 1999) and proportional thinking (Lawton, 1993). Research in mathematics and science education reveals that it is difficult for students to apply their mathematical knowledge in different contexts (Delice & Roper, 2006) as the context requires an additional layer of the meaning to variables in the mathematical expression in order to be applied in the science context (Dray & Manogue, 2005), and particularly in integration problems (Alves et al., 2019; Jones, 2015a).

Carlson et al. (2003) developed curricular materials aiming to promote students' understanding and reasoning abilities about the FTC, which contained tasks given in an extra-mathematical context whenever possible. Twenty-four students were assessed for their pre-calculus concepts at the beginning of the term and after instruction. The extra-mathematical contexts used for the assessment were water filling up a tank and distance traveled over time. The RoC function was provided graphically in the tank context, and algebraically in the distance context. A third item used a geometry context – a circle expanding in size. The questions asked in each item were related to the FTC and the accumulation function. The results showed that most students have completed the course with a strong understanding of notational aspects of accumulation, as defined by the researchers. Such understanding includes that the notation $F(x) = \int f(x)dx$ means that F (which represents an accumulation function) is an antiderivative of f, and that f is the RoC of F.

1.4. Abstraction in Context

Abstraction in Context (AiC) is a theoretical framework proposed by Hershkowitz et al. (2001) for studying learners' construction of new (to them) abstract mathematical knowledge. The knowledge intended by the designer or teacher to be constructed is analyzed a priori into knowledge elements that include concepts, procedures, and strategies. In this paper, the relevant part of the a-priori analysis will be presented in section 4 as part of the results since the design of a learning activity and the analysis of the structure of the content forms an integral part of the research. Learners' processes of constructing these knowledge elements are then analyzed by means of three observable epistemic actions: Recognizing (R) – the learner identifies a previous construct as relevant to the task at hand; Building-With (B) – the learner uses a recognized construct for achieving a local goal, and Constructing (C) – a new construct emerges for the learner by recognizing and building-with previous constructs. As R-actions are nested in B-actions and R- and B-actions are nested in C-actions, Hershkowitz et al. (2001) proposed the name "dynamically nested RBC-model". The model and its use are described in more detail by Dreyfus et al. (2015).

2. Rationale and research questions

The research presented in this paper approaches integration via accumulation; it aims to give secondary students, who have not yet learned integration, an opportunity to develop ways of thinking that will be useful in taking an accumulation approach to integration at school as well as in later tertiary level studies. Such thinking will be called in this research *Accumulative Thinking*.

We define Accumulative Thinking as a combination of specific knowledge and its application:

1. Awareness of the nature and the multiplicative structure of the "bits" that are accumulated, as well as the dynamism of the process of accumulating these bits.
2. The ability to apply this knowledge, for example, to be able to use it for reasoning about some characteristic of an accumulation function, such as its concavity, when the RoC function is given graphically.

This paper focuses on the following questions:

1. What is the structure of Accumulative Thinking?
2. How do students construct the elements of Accumulative Thinking?

Research question 1 will be answered by an a-priori analysis of knowledge elements that constitute Accumulative Thinking. This will result in a set of 16 knowledge elements, out of which 10 knowledge elements that are relevant to this report are presented in section 4. To answer research question 2, we will describe how students construct these 10 knowledge elements (section 5).

The investigation of how students construct knowledge about accumulation (as being dependent on a process that is being carried out on another function) is expected to help improve the pilot design introducing accumulation presented below (section 3.2).

The teaching approach we design for the secondary level is expected to contribute to the transition to the tertiary level, and to serve as intuitively acceptable and meaningful base at the tertiary level. We expand on this in section 6. This is in line with recent claims that the secondary-tertiary transition is neither continuous nor discontinuous but both (Gueudet et al., 2016).

Carlson et al. (2003) used the accumulation approach to develop tertiary students' understanding of the FTC (see sections 1.2 and 1.3); they presented problems using various extra-mathematical contexts. Our research presents the accumulation approach in the context of filling a pool; however, our mathematical focus is different – it is to develop Accumulative Thinking in secondary students, so it can serve later as building blocks required for introducing the integral concept, including the FTC.

3. Methodology

3.1. Research design and population

A learning activity was designed and piloted with a pair of grade 11 students who study mathematics at an advanced level. The pilot gave insight into changes required, and the learning activity was re-designed into a final version for this research.

In this research 6 students from two different schools participated; they all learn mathematics in grade 11 at the advanced level. When they participated in the research, the students had already learned the topic of differentiation but not yet that of integration.

The students carried out the learning activity in pairs and were asked to collaborate and discuss the tasks they were working on. However, each student had their own learning activity sheet, which had space for answers, and the students were instructed that in case there is a disagreement, each student will write their own answer. The researcher (the first author) presented questions to the students only to clarify their utterances and the mathematical meanings behind the course of action they took.

The interviews were audio-recorded and transcribed, and the transcriptions were analyzed, together with the learning activity sheet of each student, using the RBC model. The goal of this analysis was to gain insight into the learning processes of the students.

3.2. The learning activity

The learning activity was designed following Tabach et al. (2008). To allow a smooth transition from secondary to tertiary studies of the integral concept, the activity we designed introduces accumulation to high-school students in an elementary manner. The activity uses the context of water flowing into a pool and deals with accumulation by leading the students to consider the bits that accumulate, their structure as products of *time duration* × *water flow rate*, the effect the RoC function – the rate of flow of water – has on the bits, and the accumulation function as sum of the bits accumulated up to a given time. Thus, the tasks in the activity were designed with the intention of leading the learners to build the function representing the amount of water in the pool as a function of time.

The activity has three parts: the first part deals with the case of a constant RoC, the second with the case of a RoC constant in segments, and the third with the case of a linear and decreasing RoC. All RoC functions in this activity are positive since the activity aims to serve as an introduction to accumulation.

In the first part of the activity the students are given the constant rate at which a pool is being filled with water. They are given consecutive time intervals and are asked to fill in for each interval the time period, the rate of flow and the amount of water added. This task was designed to offer an opportunity for the students to conclude and use the multiplicative relationship of *time duration* × *water flow rate* = *amount added*. The students are then asked about equal bits, meaning time intervals in which the same amount was added. The students are asked to find the accumulated amount for various points in time, based on the amounts they calculated previously. Next, the students are asked to sketch the graph of the accumulation function of the given constant RoC. In the next question the students are given a GeoGebra animation where they may select start and end times. When executing the animation, the rectangular area under the graph and above the time axis is being filled in from the starting point and grows continuously to the end point of the selected time interval. The students are asked questions about the graphical representation of each of the elements in the above multiplicative relationship; they are offered repeated opportunities to conclude that the area of the rectangle above a time interval represents the amount of water added in that time interval, and to connect the graphical representation with the numerical representation of the multiplicative relation *time duration* × *water flow rate* = *amount added*. The animation aims to help the students grasp visually the process of accumulation as a dynamic one. Screen shots of this animation are provided in Figure 4 (section 5.4).

In the second part, which deals with the case of a RoC which is constant in segments, the students are given two graphs, both constant in segments, representing the water flow rate. The students are asked to draw the accumulation function for each water flow rate and find the amount of water that was added in a given time interval, during which the rate of filling up the pool changes. The amount added can be calculated since the axes show units and values. In the following question, the students are given two graphs, both constant in segments, on two identical coordinate systems without units nor values on the axes. They are asked to determine if the amounts of water accumulated in the pools are equal or not. This last part was designed to give the students another opportunity to express that the area under the graph represents the amount of water when the amounts accumulated cannot be calculated since the coordinate axes are not labeled with units and values. This second part also deals with tiny bits, where they are asked to calculate a bit by splitting it to two sub intervals, where the second one is much smaller.

In the third part, the students are presented with the graph of a linear and decreasing RoC, representing the water flow rate. They are asked about the amounts that are added over time and requested to draw a sketch of the accumulation function giving them an opportunity to conclude that the accumulation function is concave downward.

3.3. Data analysis

The data analysis consists of two stages: An a priori analysis and an analysis of the interview data. The a priori analysis examines the learning activity in view of the students' previous knowledge. Its aim is to identify the knowledge elements the designer of the learning activity intended the students to construct while carrying out the activity. These knowledge elements are defined operatively. The a priori analysis is presented in section 4.

The data from the interviews were analyzed according to the methodology of AiC. First the interviews were transcribed and presented in a table which contains the turn number, the speaker, the utterance and a fourth column for the epistemic actions. The RBC model was used to identify epistemic actions of recognizing (R) a knowledge element, building-with (B) a knowledge element and constructing (C) a new knowledge element. These epistemic actions were marked along the transcription. The RBC analysis is presented in section 5; a transcript with epistemic actions is included in section 5.6.

4. The structure of Accumulative Thinking

The a priori analysis of Accumulative Thinking as defined in section 2 resulted in a list of sixteen knowledge elements intended to be constructed by the students; together, these knowledge elements constitute Accumulative Thinking. Each of them has been given an operative definition, allowing the researcher to assess whether a student has constructed the knowledge element (see Table 1). In addition, we identified fifteen preliminary knowledge elements, assumed to have been constructed earlier. For the economy of space, we only present an overview of the preliminary elements; among the 16 main knowledge elements, we only present the 10, which are relevant to this paper, followed by a table with their operative definitions. These knowledge elements will be illustrated in section 5, in parallel with the analysis of the students' work. This illustration includes two of the preliminary knowledge elements.

Following the learning activity, the first few knowledge elements relate to the case where the RoC is constant. The accumulating bits have a multiplicative nature since the amount added in a time interval is the product of the length of the time interval by the rate at which the quantity accumulates (knowledge element M_nr: Multiplicativity – numeric representation). The bit has a graphical representation as the corresponding rectangular area (knowledge element M_gr: Multiplicativity – graphical representation). The ratio of the amounts of water entering the pool in two different time intervals equals the ratio of the lengths of the time intervals (preliminary knowledge element P6 - proportion).

In the case of a RoC that is constant in segments, the accumulation function is linear in segments (AFCS), and the area under the RoC graph represents the amount of water added (A_cs). In all cases, the accumulation process is dynamic, meaning that the accumulation function gives the amount accumulated at any given time (AF). Summing up consecutive bits within a given time interval results in the amount accumulated in that entire time interval (S: Summing consecutive bits). Conversely, the amount that accumulates in a time-interval can be split into two sub intervals, where the second one is much smaller than the first sub-interval, for example, 10 times smaller (TB_r: Tiny Bit reduction). In the case of a linear and decreasing RoC, the multiplicative connection *time duration × water flow rate = amount added* cannot be used as is because the rate does not have a constant value. The idea of instantaneous RoC is not appropriate for students at this stage of learning. Therefore, it becomes imperative to use the area under the graph as representing the amount added (A_dl: Area – decreasing and linear RoC). The bits that accumulate are graphically represented by the trapezoids, formed by the graph and the time axis within a given time interval. As time increases, the amounts added (the bits) with the same time duration are getting smaller (DB: Decreasing Bits). Hence, as the RoC decreases, the accumulation function in this case is concave downward (AFDL: Accumulation Function of a Decreasing Linear RoC).

Table 1. Knowledge Elements related to this article and their operative definitions

Knowledge Element	Abbreviation	Operative Definition
Multiplicativity - numeric representation	M_nr	In the case of constant RoC, the student expresses the multiplicative connection between the quantities: time duration × RoC = amount added.
Multiplicativity - graphical representation	M_gr	In the case of constant RoC, the student expresses that the amount of water added equals the area under the graph in a given time interval.
Accumulation function of a RoC constant in segments	AFCS	The student draws the graph of the accumulation function as continuous and linear in segments and describes the amount added in a time segment using M_nr or M_gr.
Area under the graph in case of a RoC constant in segments	A_cs	The student expresses that the total area under all segments up to the current

Knowledge Element	Abbreviation	Operative Definition
		time represents the amount accumulated.
Accumulation Function as a dynamic process	AF	The student expresses that as time flows water is being accumulated, and that at any given time the accumulation function gives the amount that has been accumulated from the starting point in time to the given time.
Summing consecutive bits	S	The student expresses verbally, graphically or otherwise, that the quantity of water added in a time interval equals the sum of the quantities added in its consecutive time sub-intervals.
Tiny Bit reduction	TB_r	The student splits a given time interval into 2 consecutive sub-intervals, the second of which is much smaller than the given time interval, and then the student sums the two quantities added in the sub-intervals to calculate the amount added in the given interval.
Area – decreasing and linear RoC	A_dl	In the case of a linear decreasing RoC, the student expresses that the area of the trapezoidal geometric shape bounded by the graph in a time interval represents the amount added in that time interval.
Decreasing Bits	DB	The student constructed A_dl and expresses that since RoC is decreasing the amounts added in time intervals with the same Δt are getting smaller because the geometrical shapes (representing the bits) are getting smaller.
Accumulation Function of a Decreasing Linear RoC	AFDL	The student constructed A_dl **and** uses DB to justify the downward concavity of the accumulation function.

5. Students construct the elements of Accumulative Thinking

In this section, we present empirical evidence of construction processes of the students Ana & Zoe, Roy & Don, and Tim & Nic, that contribute to Accumulative Thinking. We use *italics* to mark the epistemic actions *recognizing*, *building-with* and *constructing*. Detailed RBC analyses of all transcripts of the three pairs of students have been carried out as explained in section 3.3. Here we present summaries of at least one construction process for each knowledge element; for one of them we demonstrate in detail how the analysis has been carried out by showing a partial transcript and marking the epistemic actions for relevant utterances in section 005.6. Tiny bit reduction (TB_r)(Table 2).

5.1. Multiplicativity – numeric representation of a bit (M_nr)

The activity starts with a pool being filled with water at a constant rate of 30 liters per minute, which is described verbally. The students are given a table with consecutive time intervals and are asked to fill in the length of the interval, the rate at which the pool is being filled and the amount of water that is added in each time interval.

Roy and Don express the numeric representation of a bit by multiplying the time duration (second row, **Figure 1**) and rate of filling (third row) in each time interval to get the amount of water that was added (fourth row). They also verbally describe the multiplicative connection in their written answer. Hence they *constructed* M_nr.

קטע זמן*	[0, 1.4]	[1.4,2.0]	[2.0,2.6]	[2.6,2.7]	[2.7,2.8]	[2.8,2.9]
משך זמן מילוי (דקות)	1.4	0.6	0.6	0.1	0.1	0.1
קצב מילוי (ליטר בדקה)	30	30	30	30	30	30
כמות מים שנוספה (ליטר)	42	18	18	3	3	3

Figure 1. Don's table. Rows' titles: 1st row – Time interval, 2nd row – Time duration (min), 3rd row – Rate of filling (liters per min), 4th row – Amount of water added (liters)

Ana and Zoe, however, don't express the numeric representation of a bit, but rather *recognize* and *build-with* preliminary knowledge element P6 (the ratio of the amounts of water flowing in two different time intervals equals the ratio of the lengths of the time intervals) in order to find the amount added in each bit. For example, to find the amount added by time interval [1.4,2.0], they first convert the time duration from minutes (0.6) to seconds (36), then they calculate the amount added in 36 seconds, as shown in **Figure 2**, by *recognizing* and *building-with* P6.

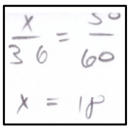

Figure 2. Using proportion to find bits by Ana and Zoe

In summary, while working on the described task, Roy and Don constructed the knowledge element M_nr, while Ana and Zoe did not.

5.2. Summing up consecutive bits (S)

Summing up consecutive bits within a given time interval results in the amount accumulated in that entire time interval (knowledge element S). In the first part of the activity, dealing with a constant rate of flow, the students are asked to find the amount accumulated up to various points in time (which correlated to the consecutive bits shown in **Figure 1**). To do so, all three pairs sum up the amounts of the consecutive bits up to the required one, and fill in the table, expressing *construction* of knowledge element S. **Figure 3** shows Tim's calculations, filling in the second row the total amount of water that accumulated in the pool up to the given minute in the first row.

עד דקה	1.4	2.0	2.6	2.7	2.8	2.9
כל כמות המים שהצטברה בבריכה (ליטר)	42	42+18 = 60	60+18 = 78	78+3 = 81	81+3 = 84	84+3 = 87

Figure 3. Finding the accumulated amount by Tim.
Rows' titles: 1st row – Up to minute, 2nd row – Total amount of water that accumulated in the pool (liters)

5.3 Initial thinking about accumulation (starting point of constructing the accumulation function as a dynamic process – AF)

At the end of the first part of the activity, dealing with a constant rate of flow, the students are given, for the first time, a definition of an accumulation function: *y=A(t) is a function that represents the amount of water that accumulated in the pool from minute 0 to minute t. It is called an accumulation function.* They are then asked to draw the accumulation function for which they calculated the accumulated amounts (see 5.1 and 5.2) and are asked why it is called an accumulation function. All pairs drew a correct increasing straight-line graph; as to why it is called an accumulation function, Zoe wrote:

> The value of *y*, which represents the amount of water, increases based on the preceding *y* value. Consequently, the water accumulates and rises without any instances of descent due to accumulation.

Roy (from a different pair) wrote:

> Because the amount of water being added in each minute is constant, in every point on the function, the *y* value accumulates and increases at 30 liters per minute.

While Zoe uses the pool context to explain the increasing property of the accumulation function graph she drew, Roy uses the slope of the accumulation function graph to explain why it increases. It is interesting, though, that both students choose to explain why the accumulation function graph is increasing when answering why the accumulation function is called that way. The students' answers mark the starting point of constructing AF since both are describing (although in different ways) the process of the accumulation of water in the pool.

5.4 Multiplicativity – graphical representation of a bit (M_gr) and Accumulation Function as a dynamic process (AF)

Here we describe how students construct knowledge elements M_gr and AF when working on the task with the GeoGebra animation (section 3.2). The process of constructing of AF started prior to this task (section 5.3).

In the first part of the activity, which deals with a constant RoC, the students are requested to select a time interval (start and end) and play the animation, where the horizontal axis is time and the vertical axis is the flow rate. During the animation, the area under the graph and above the time axis is being painted, from the starting point and growing continuously to the end point of the time interval. As time flows, the painted area increases. In **Erreur ! Source du renvoi introuvable.** there are 3 screen shots from the animation on the time interval [1.5,1.9]. The area is colored in steps of 0.1 on the time axis, which makes the coloring process look continuous.

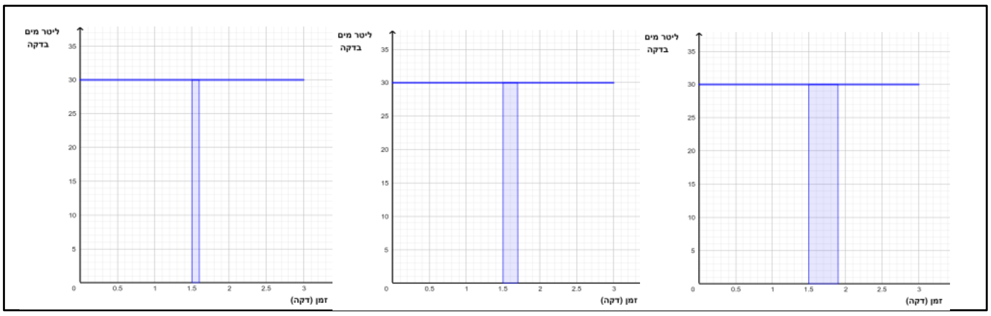

Figure 4. Screen shots from the GeoGebra animation

5.4.1. Accumulation function as a dynamic process (AF)

After playing the animation as described above, the students are asked what the animation illustrates.

Roy and Don discuss the animation:

Roy: I understand, so the graph represents the rate… the amount of water…
Don: Which was added…
Roy: That is being added…
Don: In the given period of time.
Roy: In the given period of time, not in relation to the given period time, because…
Don: In the given period time, period of 0.6 second.
Roy: Is being added to the pool… ahh yes, is being added in the period of time… no, in…
Don: The period of time…

Roy and Don describe a dynamic process. The use of a progressive time "is being added", for example, indicates that.

Roy then writes the following answer:

> The painting of the graph in the animation illustrates the amount of water that is being added to the pool by the area that the graph forms with the time axis.

We interpret Roy's answer as expressing an evolution as he visually grasps the accumulation as an ongoing process. Roy ties the painting of the increasing rectangular area in the animation, which looks continuous, with the accumulation of water, suggesting that Roy "sees" the accumulation as a continuous process. This interpretation gets affirmed later when the students draw continuous functions, and hence shows the *construction* of AF.

Ana and Zoe also discuss the animation:

Ana: That's right, but imagine you have a pool, you don't have 30 liters of water straight away.
Zoe: Right, it's gradually.
Ana: Right, it's gradually, like, that's what I had in mind.
Zoe: I don't get it.
Ana: Imagine we have like… such a container…
Zoe: And you fill it.
Ana: We want to fill it with 30… umm… 30 liters, we take a pipe, and we don't fill it straight away.
Zoe: That's right, it is done gradually.

Ana uses the context of the pool that is being filled with water and describes the process of accumulation as a gradual one, which seems to help her grasp the accumulation process as dynamic and continuous. As in Don's case, this interpretation gets affirmed later when the students draw continuous functions, and hence express the *construction* of AF.

5.4.2. Multiplicativity – graphical representation of a bit (M_gr)

The bit that accumulates has a graphical representation namely that the amount accumulated by that bit is represented by the rectangular area of the bit (knowledge element M_gr).

After playing the animation, the students are asked where in the animation they can see the rate, the time period, and the amount of water, to draw the amount of water that was added during time interval [1.5,1.9] and to explain why their drawing represents the amount of water added in that time interval.

Ana marks the horizontal segment length of 0.4 as the time period, the rate as vertical segment of 30 and the area of the rectangle, as shown in **Figure 5**, and answers:

> Because the time duration that passed times the filling rate equals the amount of water that was added– which is the rectangle area.

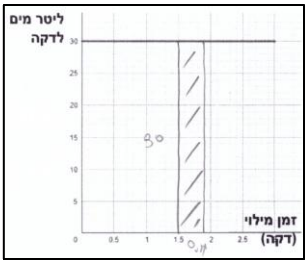

Figure 5. Ana's drawing the amount of water

As mentioned in 5.1, Ana hasn't constructed M_nr (numeric representation of a bit) up to this point. In her answer above Ana expresses the *construction* of both the numeric representation of the bit (M_nr) by explaining that the amount of water is the product of the time and rate, and the graphical representation of the bit (M_gr) by connecting the rectangle area as representing also the amount of water.

Similarly, Roy who has already constructed M_nr earlier, marks the segments (**Figure 6**) in a similar way to Ana's, and describes that the area of the rectangle represents the amount of water, expressing the *construction* of the graphical representation of a bit (M_gr):

> Because the illustration gives a rectangular shape that has an area that represents the relation between the rate of filling and the time of filling, whose product gives the amount of water that was added.

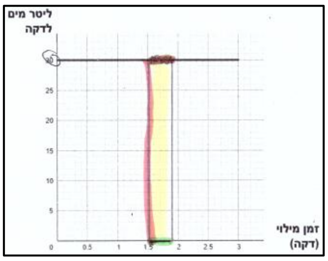

Figure 6. Roy's drawing the amount of water

5.5 Accumulation function of a RoC constant in segments is linear in segments (AFCS)

In the second part of the activity, the students are required to draw the function representing the amount of water in the pool when the rate of flow is constant in segments (**Figure 7**), where again the horizontal axis is time (minute) and the vertical axis is the rate of filling (liters per minute). All pairs split the time according to the segments given in the rate of flow graph, and then used the numeric representation of the bit to calculate the bit added in each segment and sum the consecutive bits to get the total amount accumulated, as demonstrated by Tim in **Figure 8**.

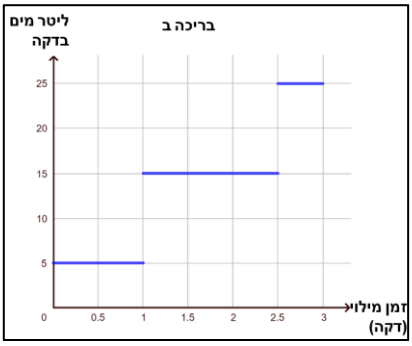

Figure 7. Rate flow of a pool

All three pairs sketched the accumulation function as a graph of straight-line segments as in **Figure 8**. Hence the students have *constructed* knowledge element AFCS (accumulation function of a RoC constant in segments) by *recognizing* and *building-with* knowledge elements M_nr and S.

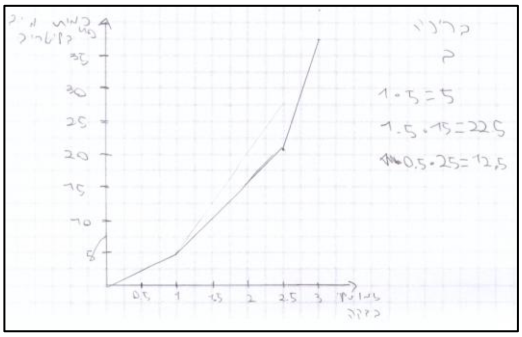

Figure 8. The accumulation function sketched by Tim

5.6. Tiny bit reduction (TB_r)

The amount that accumulates in a time-interval can be calculated by summing the amounts added in its two sub-interval, when the second sub-interval is much smaller than the first sub-interval (knowledge element TB_r). In the first part of the activity, which deals with a constant rate of flow, the students are asked to calculate the amount of water accumulated up to a certain point in time by using the amount accumulated up to an earlier point in time: (1) calculate the accumulated amount up to minute 1.43 using the amount accumulated up to 1.4; (2) calculate up to minute 1.432 using 1.43; (3) calculate up to 1.433 using 1.432.

Table 2 presents a part of the conversation between Roy and Don discussing task (1) with RBC analysis.

Table 2. Discussion between Roy and Don with RBC analysis

Turn	Speaker	Utterance	RBC
121	Don	Minute 1.43… the amount of water accumulated is… ahh, I know, he multiplied, 1.4 times… three... subtracting this from this [using the calculator]	R M_nr R&B P14
122	Roy	How does he know?	
123	Roy	Wait, one second… if in each… in… it is strange to me.	
124	Don	If in 1.4 he did 42 (liters), like, it was filled with 42 (liters)…	B M_nr
125	Roy	Ahh, he knows the rate of [filling-up] the pool, yes.	

126	Don	So he is subtracting this from this… he got 0.9.	B P14
127	Roy	Yes yes yes. One second. I know the rate of filling up the pool? Good. [reading the question] *he used the amount of water accumulated up until… to find the amount of water accumulated up until…* he did 1.4 time 30 plus 0.03 times 30.	R&B S R&B M_nr C TB_r

Roy and Don first split the bit into two smaller bits. They *recognize* and *build-with* preliminary knowledge element P14 (time period – the difference between the start and end time of a time interval gives time period of the time interval) to get the time length of each bit. They then use the numeric representation to calculate the amount added in each split bit, hence *recognize* and *build-with* M_nr. They sum the amount added in both split bits (*recognize* and *build-with* S), thus expressing the *construction* of TB_r (Tiny Bit reduction). In their way of solving the pair did not use a previously calculated amount (up to minute 1.4), as they were instructed, but rather calculated it again. In the following questions, which followed 2 and 3 (as described above), they did consider the amount that was already calculated as they were instructed. For example, in (3), in order to calculate the amount of water added up to minute 1.433 by using the amount of water added by minute 1.432 (which they have already calculated to be 42.96), as can be seen in **Figure 9**.

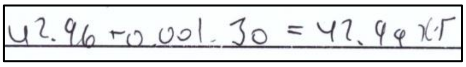

Figure 9. Tiny bit reduction by Roy

Ana and Zoe, also express the construction of TB_r (Tiny Bit reduction); however, they did not recognize nor build-with M_nr, but rather, they again (see section 5.1) *recognize* and *build-with* preliminary element P6 (proportion) as can be seen in **Erreur ! Source du renvoi introuvable.**.

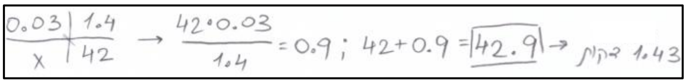

Figure 10. Using quantities proportion in Tiny Bit reduction by Ana and Zoe

5.7. Area under the graph represents the amount of water added to the pool in case of a RoC in segments (A_cs)

In the case of a constant RoC, the bit that accumulates has a graphical representation namely that the amount accumulated by that bit is represented by the rectangular area of the bit (knowledge element M_gr); the process of constructing this knowledge element was described in section 5.4. This knowledge is naturally generalized to a RoC that is constant in segments: the area in a time interval under the RoC graph and above the time axis represents the amount of water accumulated in this time interval.

In the case of a RoC constant in segments, the students were given two graphs, which are constant in segments, representing the rate of flow in pool A and in pool B in separate coordinate systems. Both coordinate systems are empty but have identical units. The students were asked how they could determine whether the amounts accumulated in the two pools are equal or not. Ana counted 18 squares under the graph representing the rate of filling of pool A and above the horizontal axis, and 16 squares under the graph representing the rate of filling of pool B, as shown in Figure 11, and determines that the amount of water accumulated in pool A is greater than in pool B. Ana then explains that the area under the graph represents the amount of water that was accumulated, hence expressing the *construction* of A_cs – that the area under the graph in case of a RoC which is constant in segments represents the amount of water accumulated.

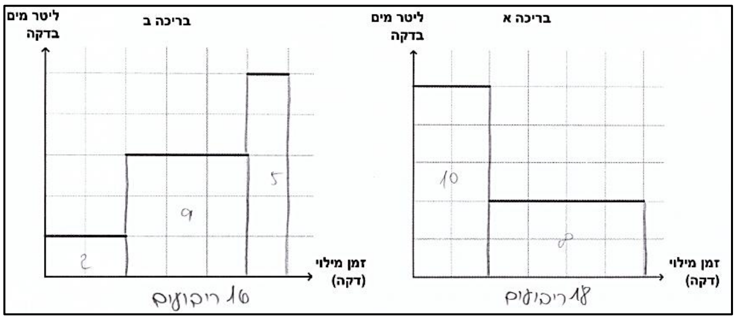

Figure 11. Counting the squares under RoC graph to compare the accumulated amounts by Ana

5.8. Area under the graph represents the amount of water added to the pool in case of a linear decreasing RoC (A_dl)

In the third part of the activity, the students are given a graph representing a linear and decreasing rate at which a pool is filled with water (Figure 12).

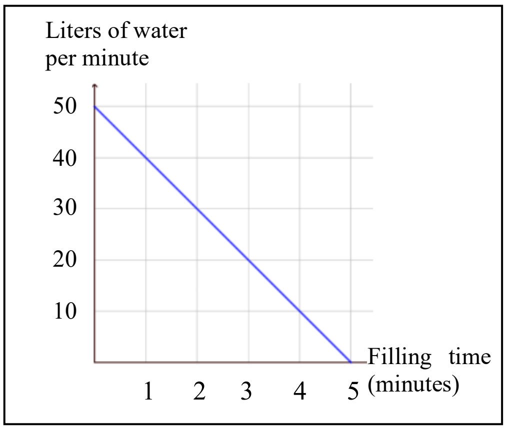

Figure 12. Linear and decreasing rate of flow

The students Don and Zoe (from two different pairs) first exhibit a chunky way of thinking by suggesting splitting the time to 1-minute bits, and multiply the rate (at the left border of the interval) with the time in order to get the accumulated amount, meaning they recognized M_nr and S. However, their partners Roy and Ana (separately in each pair) said that this is not applicable since the rate of flow is not constant. In order to draw the accumulation function, each pair then split the time into 1 minute length bits, calculating the area of the respective trapezoid, hence expressing the *construction* of A_dl and *recognizing* and *building-with* it. They also sum up the consecutive bits to get the accumulated amount, hence *recognizing* and *building-with* S. Ana's way of doing this is shown in detail in **Figure 13** where the horizontal axis represent the time of filling (min) and the vertical axis represents the amount of water (liter).

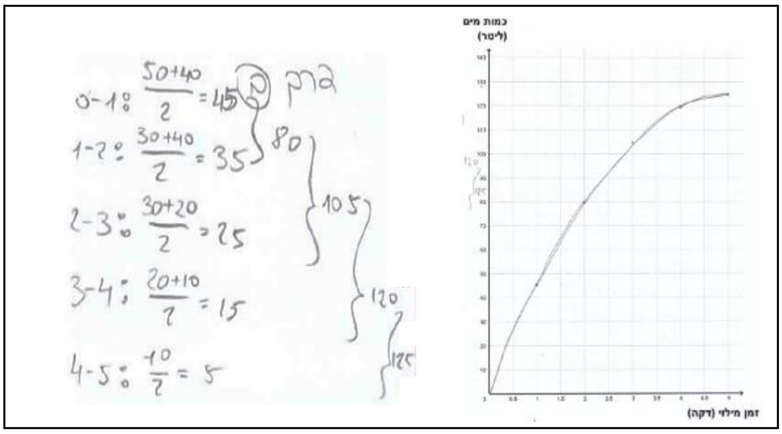

Figure 13. Ana's accumulation function

5.9. In the case of a linear and decreasing RoC, the bits are decreasing (DB) and the Accumulation function is concave downward (AFDL)

In the third part of the activity which deals with linear and decreasing RoC (**Figure 12**), the students are also asked what can be said about the amounts of water that are being added to the pool. Don and Roy answered that the amounts are getting smaller and when they are asked by the researcher to elaborate, Don says:

> It can also be seen according to the area of each part. Of each moment…by the area of the graph at any moment. In the first second the area is the largest, and in the 2nd second the area starts to get smaller.

When he was asked by the researcher to show the areas that he refers to, he referred to the graph representing the rate of filling as a function of time and pointed out the areas that were marked by his partner, Roy, as I, II, III, IV, VI as shown in **Figure 14**.

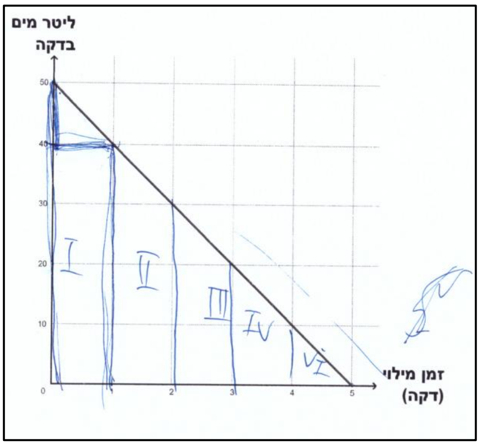

Figure 14. Numbering the bits by Roy

In his answer, Don expressed the construction of DB (in case of a decreasing RoC, the amounts added with the same time interval are getting smaller). The process of constructing DB occurs by *recognizing* and *building-with* previously constructed knowledge element A_dl (the area under the graph in case of a linear decreasing RoC represents the amount added). He then *recognizes* and *builds-with* DB to explain why the accumulation function is concave downward, hence *constructing* AFDL.

To answer why the accumulation function is concave downward, Nick said: "The rate at which the pool is being filled decreases, resulting in a decrease in the amount of water added to the pool". To answer the same question, Ana explained that the graph is concave downward since the amounts that are being added are gradually decreasing. In her answer Ana expresses the two components of Accumulative

Thinking: she expresses knowledge about the bits, describes the accumulation process as dynamic and applies both to draw the graph (Figure 13).

6. Discussion

The learning activity designed for this research aimed to develop Accumulative Thinking (see section 2). The analysis of the students' work allows us to argue that the goal was met (see section 5). In this section, we will discuss the various properties of the learning activity and its approach to integration that we believe may help smooth the transition from secondary to tertiary studies on the subject of integration, based on the analysis of the students' work.

6.1. Adopting RoC as the didactical base and using the pool context

As mentioned, the didactical base for learning accumulation can be viewed as 'adding up pieces' or as derived from RoC (Ely & Jones, 2023). In this research, we adopted the latter approach. RoC is a crucial concept for integration and is formally taught at the tertiary level. However, it is a challenging concept for secondary students. To address this, the activity proposed here uses an extra-mathematical context (see section 1.3) of a pool being filled with water. When calculating the amount of water added with a constant RoC, Ana and Zoe did not multiply the time duration by the rate. Instead, they used their perception of water accumulating at a constant rate and its proportional property: the ratio of the amounts of water flowing in two different time intervals equals the ratio of the lengths of the time intervals (see section 5.1, Figure 2). The pair circumvented the difficult notion of RoC by relying on their understanding of the context to find the amount of water added in various time intervals.

We chose the pool context because we believe that in this context high school students can naturally grasp the accumulation process. The analysis suggests this hypothesis has been verified. For example, Zoe wrote, "…Consequently, the water accumulates and rises without any instances of descent due to accumulation"; observing a pool being filled with water, Zoe sees the process of accumulation through the rising water level. This contrasts with the context of distance over time, where the accumulation of distance is not tangible, making it harder to visualize its dynamic and continuous process. As for RoC as a didactical base, Elias et al. (2023) also found evidence that the pool context helps students grasp RoC intuitively, enabling meaningful action and reasoning.

Another reason for using an extra-mathematical context is the prior knowledge of secondary students. Not using such a context would require a formal mathematical context, presenting the RoC function as a derivative function. Introducing the term 'derivative' might lead secondary students to apply their existing knowledge about the connection between a function and its derivative, diverting their focus from the accumulation process. Our design decision addresses the gap between secondary and tertiary levels, serving both levels. It enables the use of the RoC concept in an informal, context-dependent manner at the secondary level, supporting later formal studies at the tertiary level.

6.2. The relation between a RoC function and its accumulation function.

The accumulation approach has been developed within the framework of quantitative reasoning (Thompson, 2022), relating directly to the meaning of the integral in contextualized situations, such as the pool used in this research. Most high school students will encounter integrals mainly in contextualized situations during their later studies. Only the few who major in pure mathematics will focus on the integral as an abstract mathematical object.

The accumulation function is constructed from a given RoC function, approximating its antiderivative, specifically the antiderivative whose value at the beginning of the accumulation process is zero. In other words, the derivative of the accumulation function approximates the given RoC function. Hence, the approach includes the Fundamental Theorem of Calculus (though this research did not explicitly address that issue). Therefore, our approach offers the advantage of a single notion of integral as accumulation function and its value, rather than two notions, definite and indefinite integrals, which are only vaguely connected for most of high school students (see section 1.2).

6.3. Riemann sums and accumulation

At the university level, integrals are usually approached via Riemann sums. In many countries, this approach is considered “beyond the students” at the secondary level, exemplifying Klein’s (2007) first discontinuity between secondary and tertiary levels. The accumulation approach to integration is closely connected to Riemann's definition of the integral as the limit of a sum of products. If the variable of integration is time, the products are of the form time duration × RoC in that interval, giving the 'bits' that accumulate. Accumulative Thinking as introduced here includes these accumulated bits and the dynamism of the accumulation process.

6.3.1. Sums of products

Throughout the learning activity, students dealt with summing products of time intervals and positive RoC (see section 5). Given a positive RoC, the product resulted in a positive value. In the case of a negative RoC, the product results in a negative value. Therefore, the students’ success in constructing knowledge in the case of a (positive and) decreasing RoC, suggests that knowledge about the numerical and graphical representation of a bit might also help overcome difficulties when dealing with a negative RoC (see section 1.2), aiding in understanding why a definite integral of a negative function results in a negative value and the relationship of this value with the area bounded by the function and the x-axis. Promoting the understanding of the sum of products may prevent students from acquiring only formal techniques, as reported by researchers (e.g., Thompson & Harel, 2021).

Although accumulation is a basic notion in daily life, thinking about accumulation is challenging for students since they struggle to conceptualize the bits being accumulated (Thompson & Silverman, 2008). Moreover, understanding the accumulation function is not trivial, as its values depend on those of another function, the RoC function. To address these difficulties, the activity asked the students to calculate the accumulated amounts for time intervals of varying size supporting them in thinking flexibly about the added amounts, (see sections 5.1 and 5.6). They summed the bits to get the total accumulated amount for various RoC functions (constant, segmented constant, and linear and decreasing RoC) and built an accumulation function by referring to the accumulated amounts as the function's values. Introducing secondary students to accumulated amounts and the accumulation function in an accessible way supports their later tertiary studies. Additionally, describing the accumulation process as a sum of products can prepare secondary students for the introduction of Riemann sums at the university level.

6.3.2. The dynamic nature of the accumulation process

To address the dynamic nature of the accumulation process, the design included tasks requiring students to calculate the accumulated amount in reduced time intervals (see section 5.6). To help students grasp the dynamic and continuous nature of the accumulation process, the design used the context of a pool being filled with water and offered an animation illustrating the accumulation on a given RoC graph. This animation, serving as a visual demonstration at the secondary level, proved useful and efficient, aligning with previous research findings (Monaghan et al., 2019). This efficiency is supported by the analysis; for instance, when discussing what the animation represents (section 5.4.1), Ana said, "… imagine you have a pool, you don’t have 30 liters of water straight away." Observing a pool being filled with water, Ana sees the accumulation process as dynamic and continuous. Similar animations could play a more sophisticated role at the tertiary level as a "grounding metaphor," relating a target domain within mathematics to a familiar source domain outside it, creating a conceptual relationship between the two (Lakoff & Núñez, 2000).

Grasping the accumulation process as dynamic at the secondary level may prepare students for later studies, where summing an infinite number of bits with infinitesimal width is introduced at the tertiary level. It may also support the complex nature of covariational thinking, which is necessary for understanding accumulation (Thompson & Silverman, 2008).

6.4. Shifting from the accumulation function of a RoC constant in segments to that of a linear and decreasing RoC.

At the tertiary level, shifting from a RoC that is constant in segments to a varying RoC (such as a linear decreasing one) requires an understanding of limits and instantaneous rate. For a RoC constant in segments, students multiplied the constant rate by the time duration to get the accumulated amount. The chosen extra-mathematical context of a pool filled with water supports this multiplication intuitively when the RoC is constant. However, this method is not applicable in the case of a linear RoC and requires an understanding of notions not yet available to secondary students, such as instantaneous rate. This difficulty is expressed by Zoe and by Don when they suggest calculating the accumulated amounts of bits for a linear and decreasing RoC by multiplying time duration by the rate of the left border of the time interval, as if the rate were constant. To address this gap, the learning activity provided opportunities to construct knowledge about the area under the graph as a representation of the amount added (see sections 5.7 and 5.8), thus enabling the students to handle an accumulation function of a linear and decreasing RoC. Constructing this knowledge proved helpful for the students. As the analysis shows, the partners of Zoe and Don suggested a more precise procedure by using the area under the graph as a representation of the amount added to perform the calculations (see section 5.8). We argue that in this case the concept of "area beneath the graph" became an epistemological mediator for students' exploration. It seems that this mediator has a very concrete meaning for students as the graphical representation of the accumulated quantity.

7. Conclusions

The mathematics curriculum in many countries extensively employs mathematical concepts to describe real-world scenarios; our findings illustrate a reversal of this relationship. The students make use of the extra-mathematical context derived from everyday life to grasp and explain mathematical concepts. We speculate that this phenomenon may become more pronounced when students generalize these ideas to new and unfamiliar situations. For instance, when explaining negative RoC, students may employ a narrative that depicts water being drained from a pool.

Based on the findings presented in this paper, we argue that our research could be useful as a didactic and methodological approach to teaching and learning integral calculus throughout the educational continuum, from secondary to tertiary education. Informal classroom observations in an 11^{th} grade class that used the same learning activity show that students used the pool context as an anchor, applying it to pure mathematical contexts. However, it is important to note that we are currently in the early stages of classroom investigations. Further empirical studies of significant duration are necessary to validate these conjectures. These studies should explore whether the knowledge construction observed in this research generalizes to more complex flow rates, whether and how other extra-mathematical contexts aid students in developing Accumulative Thinking, and what are the effects of introductory activities like the one presented here on the long-term understanding of the integral concept.

Undoubtedly, this mission is ambitious. However, there is some optimism since an accumulation approach to integration has gained increased prominence world-wide, on the basis of an approach based on quantitative reasoning (Thompson & Silverman, 2008). This approach has been shown to be particularly appropriate for applying integration in STEM subjects (Jones, 2015b), and locally has been adopted as a guideline for integration in a new advanced level high school curriculum (Dreyfus, Kouropatov & Ron, 2021).

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