MIKE THOMAS, TOMMY DREYFUS, CAROLINE YOON

BUILDING THINKING ABOUT GRAPHICAL ANTIDERIVATIVES: THE ROLE OF INTERVAL PERSPECTIVES

Abstract. The concept of function traverses the mathematics studied at school and university and is an important transitional link between the two. We examine how an interval perspective on function may help with constructing graphical antiderivative functions. In doing so a number of important constructs needed to build such a graphical understanding are considered, along with how the students in the study linked them and built with them. In addition, some of the difficulties they faced and possible reasons for them are explained. The evidence presented shows that an interval perspective on function was important in being able to construct antiderivative functions graphically. Hence, we propose that this interval perspective on function may prove useful in helping students in transition to construct a local perspective on function. In turn we suggest what a potential path for thinking about graphical antiderivatives could look like and the kind of activities that could assist transition students along it.

Key words. Abstraction, Transition, Calculus, Graph, Antiderivative, Interval Perspective.

Résumé. La notion de fonction traverse les mathématiques étudiées à l'école et à l'université. Cet article examine comment une perspective d'intervalle sur les fonctions peut aider des étudiants à construire graphiquement des fonctions primitives. Un certain nombre de concepts importantes nécessaires à la construction d'une telle compréhension graphique sont examinés, ainsi que la manière dont les étudiants de l'étude les ont reliés et ont construit avec eux. Les résultats montrent qu'une perspective d'intervalle sur les fonctions était importante pour pouvoir le faire. Par conséquent, nous proposons que cette perspective d'intervalle sur les fonctions puisse s'avérer utile pour aider des étudiants en transition à construire une perspective locale sur les fonctions. En retour, nous voyons à quoi pourrait ressembler une voie potentielle de réflexion sur les primitives graphiques et le type d'activités qui pourraient aider les étudiants à faire la transition dans cette voie.

Mots clés. Abstraction, transition, analyse, graphe, primitive, perspective d'intervalle.

In many parts of the world, there is a shift in the way mathematics is taught in school to the way it is taught in university (Thomas et al., 2015). This has variously been described as a shift from focusing on techniques that have pragmatic value (in solving tasks) to those of epistemic value (providing insight into the mathematical objects and theories studied) (Artigue, 2002); from application of techniques to their justification and significance within a mathematical theory (De Vleeschouwer, 2010); and from problem-solving skills to more abstract, rigorous, logical deductive reasoning (Engelbrecht, 2010; Leviatan, 2008).

ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES, numéro thématique 4, p. 47 – 75.

During this transition, students often have difficulties with calculus topics (Thomas et al., 2015) such as function (Dias et al., 2008), mathematical argumentation (Farmaki & Paschos, 2007), real numbers (Ghedamsi, 2008), infinite series (González-Martin et al., 2011) and limits (Mamona-Downs, 2010; Oehrtman, 2009). Research has identified graphical antidifferentiation and its relationship with integration as a particular area of weakness for students transitioning to university (Jennings, 2011; Thomas et al., 2015). To help surmount these difficulties, Thompson and Carlson (2017) maintain it is essential that school students build quantitative and covariational ways of thinking about function since these are foundational for learning calculus. In addition, Jones (2015) demonstrated the value of a multiplicatively-based summation conception based on 'adding up pieces' in order for students to make sense of definite integrals. Both of these approaches require a specific kind of thinking about functions, namely an interval perspective.

We explore how an interval perspective about functions can support students in the transition between the more technical integration skills they usually acquire at high school and the more conceptual ones they need at university, and specifically in their understanding of the graphical antiderivative. This, in turn, can lead them to see integrals as accumulation functions, leading to the Fundamental Theorem of Calculus (Jones, 2015; Thompson & Silverman, 2008). We show that students can develop an interval perspective on function through suitable activities that help them build and make sense of key antiderivative constructs and thus assist them in the transition to tertiary study.

1. The Interval Perspective

The notion of 'interval perspective' comes from a classification of four perspectives that are thought to support versatile thinking (Thomas, 2008) about functions: pointwise (or punctual), local, global (Vandebrouck, 2011; Montoya-Delgadillo, Páez Murillo, Vandebrouck & Vivier, 2018) and interval (Thomas et al., 2017). Each perspective directs attention to units of different sizes, ranging from a single point to the whole of a function's domain. A pointwise perspective focuses on correspondences between two sets of numbers, an element and its image, whereas a global perspective allows one to recognise and compare functions, to identify global properties, or to perform transformations, such as translations. Local and interval perspectives both focus on domain intervals of different scales. A local perspective focuses on very small intervals, such as finding the rate of change of f at the point x_0 using $[x_0, x_0 + h]$ and $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. In contrast, an interval perspective focuses on larger intervals, such as the concavity of a function $f''(x) < 0$ on $x \in [a, b]$, or the average rate of change of the function on a larger interval $[x_0, x_1]$, $\frac{f(x_1)-f(x_0)}{x_1-x_0}$.

School calculus often highlights pointwise and global perspectives, whereas the teaching of calculus at university often focuses on a local perspective more often than in High School, using techniques from analysis (Fernández-Plaza et al., 2013). This often causes difficulty for students. Vandebrouck (2011) claims that when secondary school calculus shifts from an initial construction of pointwise and global perspectives to continuity or differentiability it tends to erase the pointwise and global points of view but doesn't allow students to reach a local point of view. Further, he asserts that when students are asked to solve tasks where algebraic techniques are insufficient they are then unable to develop a local perspective or to consider functions as complex objects with pointwise as well as global properties. The crucial role that function perspectives play in the secondary-tertiary transition has also been noted by Gueudet and Thomas (2020).

Although school calculus often focuses on pointwise and global perspectives, interval perspectives are useful when learning about graphical antiderivatives. Consider the graphs of two functions, 1a and 1b provided in figure 1, with their antiderivatives directly below, 1c and 1d. A student may be able to recognise, globally, that each graph is similar to a parabola and hence the antiderivative should be similar to a cubic graph (this also involves recognising the parabolic nature of the graphs and recalling the form of its antiderivative). However, this is insufficient to solve the problem of drawing each antiderivative. Pointwise properties can help. For example, graph 1b has two points where the gradient is zero (at $x = 1$ and $x = 3$), while graph 1a has none. But an interval perspective provides additional information, such as intervals where the two antiderivatives have positive and negative gradients; graph 1a tells us that the antiderivative graph 1c will always be positive in gradient, whereas graph 1b provides the information that the antiderivative graph 1d will be positive on two intervals $x < 1$ and $x > 3$, and negative on the interval (1, 3).

The transition from school to university is often encumbered by a difficult transition from pointwise and global perspectives (with a focus on technical procedures) to local perspectives (to explore concepts). We examine the feasibility of using the interval perspective to help students construct and make sense of concepts of graphical antiderivative, with the hypothesis that it can act as a bridge for developing a local perspective and conceptual understanding of the graphical antiderivative.

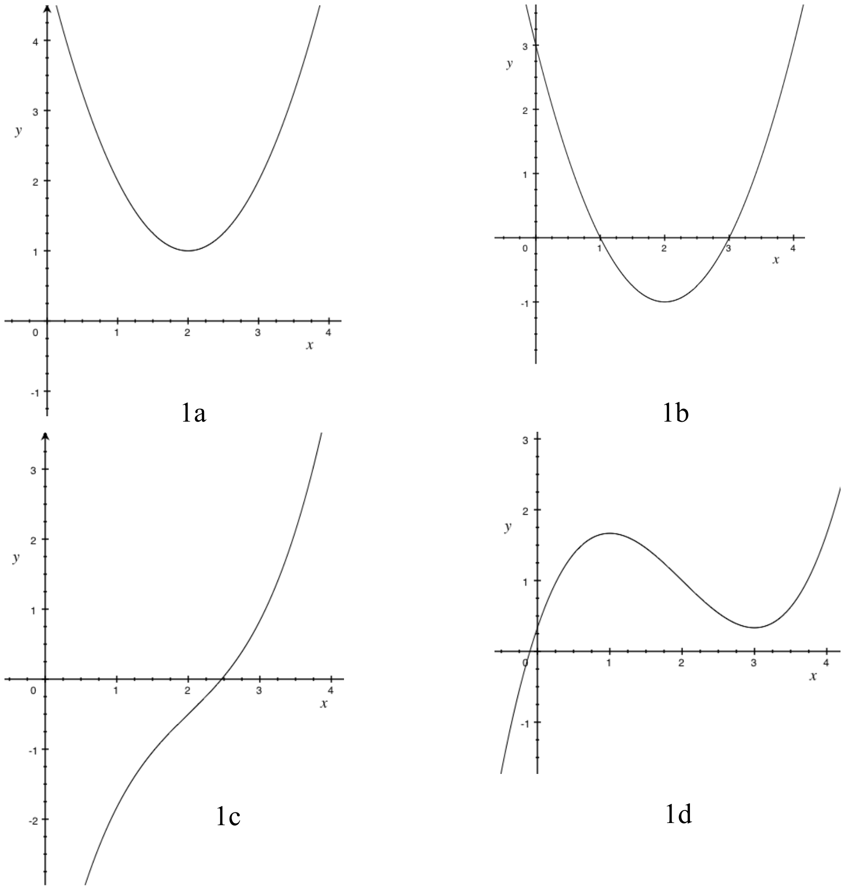


Figure 1. The need for pointwise and interval perspectives

2. Abstraction in Context

To analyse students' understanding of graphical antiderivatives, we use a socio-cognitive framework, based on Abstraction in Context (AiC), where abstraction is defined as an activity vertically reorganising previously constructed mathematical knowledge into a new structure[1], and the context includes classroom and curricular aspects of the learning environment as well as students' prior learning experiences (Hershkowitz et al., 2001; Dreyfus et al., 2015). Abstraction is a central process in

[1] This definition owes much to the pioneering work of the Freudenthal school of Realistic Mathematics Education and their development of the concept of vertical mathematisation (see e.g., Treffers & Goffree, 1985).

the learning of mathematics. From this perspective structure results from the process of abstraction.

An operational model used as a lens for observing abstraction, developed through successive iterations (Schwarz et al., 2009), comprises nested epistemic actions, and has been applied to complex processes of abstraction and environments rich in social interactions (Dreyfus et al, 2001, 2015). The model, called the RBC+C model, comprises three epistemic actions: recognising; building-with; constructing. Here recognising occurs when a student realises that a mathematical construct they are familiar with is inherent in a mathematical situation. This may be through analogy with the known construct or through specialisation, where the more general known construct is seen to have a specialised application in the situation. The action of building-with consists of combining existing constructs to attain a goal, such as solving a problem or justifying a statement. However, the central step of abstraction is the action of constructing, where existing knowledge constructs are assembled and integrated by vertical mathematisation to produce or use a new construct.

The nested nature of the epistemic actions in abstraction is such that "constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions." (Dreyfus et al., 2001, p. 310). Further Dreyfus et al. (*ibid.*) argue that abstraction passes through three phases: a need for a new structure; the emergent construction of a new construct; and finally the consolidation of the new construct through repeatedly recognizing it and building-with it in further activities (Schwarz et al., 2009). Consolidation is a long term process during which the new construct becomes freely and flexibly available. Criteria to infer the consolidation of a construct include immediacy, self-evidence, confidence, flexibility, and awareness (Dreyfus & Tsamir, 2004).

When students learn in a classroom context Hershkowitz et al. (2007) observe that it is important to address how individuals interact with other students in a group as they follow parallel processes of abstraction. In our analysis of student activity of building a graphical understanding of antiderivative functions we will employ this framework because it has proved useful to describe the construction of knowledge by small groups of interacting students and thus provides a methodological tool to examine how an interval perspective on function may arise and be employed. In order to carry out this examination, we designed a sequence of activities, and implemented it with a pair of students at the transition from school to university; we observed their process of constructing knowledge (using AiC), focusing on the emergence and use of an interval perspective when dealing with graphical tasks involving antiderivatives.

3. Method

This study is part of a larger project[2] that explored students' versatility of thinking (Thomas, 2008) as they recognise, build, construct and consolidate (Hershkowitz et al., 2001) concepts in calculus. Initially, we intended to study the thinking of students in their final or penultimate year of secondary school, but due to data collection constraints and opportunities, decided to focus on students in a first year undergraduate mathematics course in a large university in New Zealand. This undergraduate course was considered a 'bridging' course, as it was designed for students who had not completed the final year of calculus at secondary school. The course covered topics ranging from algebra and trigonometry to single variable calculus.

The two participants reported in this study, Amy and Jay, were enrolled in this 'bridging' course, and had recently finished learning about integration and differentiation when they participated in the study. Their exposure to antidifferentiation in the course was largely limited to procedure-based rules for finding antiderivatives in symbolic form. Graphical antiderivatives are generally not taught in New Zealand schools and they had not worked with them before. Hence, they were ideal subjects to examine knowledge construction in this area.

Amy and Jay volunteered to work together on antiderivative tasks in four 1-hour sessions over two weeks, outside of class time. The sessions took place in a small room where they had use of a large table space, as seen in figure 2. They were given a monetary voucher in compensation for their time, but no course credit for participating in the research. They knew each other from the bridging course from which they were recruited, but they had not previously met the researcher, who was present during each of the sessions to clarify instructions and offer encouragement but who refrained from guiding them mathematically.

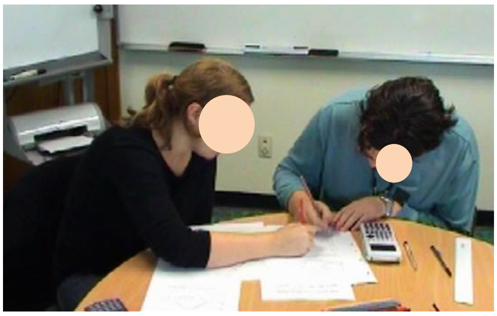


Figure 2. A room setting as Amy and Jay worked on the tasks

---

[2] Ethics approval was obtained for the research study.

Each session comprised a sequence of graph-based tasks, designed by the authors, that explored properties of graphical antiderivatives, involving drawing antiderivative functions and reasoning about local extrema, the constant of integration, concavity and points of inflection. Since the research team wished to avoid instances where students could rely on pre-learned procedures, the calculus concepts were deliberately presented graphically in the tasks rather than symbolically, with little or no numerical information given on the graph axes. The graphs were deliberately drawn so the functions did not resemble familiar polynomials and no algebraic equations were provided for the functions. The tasks were given to the students one at a time, but they had access to all previously completed tasks. The students were videotaped and audiotaped as they worked, and transcripts of their speech and actions created.

4. Results and Analysis

We present four snapshots of Amy and Jay's work in the first two sessions. In each of these we will see how an interval perspective on function enabled them to build key constructs of graphical antiderivatives. The transcripts have been analysed using the methodology of Abstraction in Context as explained by Dreyfus et al. (2015) but the full details of this analysis are not presented here in order to keep the paper reasonably short and retain the flow from one episode to the next.

4.1. Snapshot One: The emergence of a global-interval perspective

In the first session, students were presented with the distance-height graph of a tramping track (hiking trail), and were asked to plot a graph of the gradients of the track (see figure 3). This is mathematically equivalent to drawing the graph of a derivative of a function, given the graph of the function itself. This snapshot shows how Amy and Jay were able to move their thinking from a primarily pointwise perspective to an interval one, and how this provided a breakthrough in their progress towards a solution to a task.

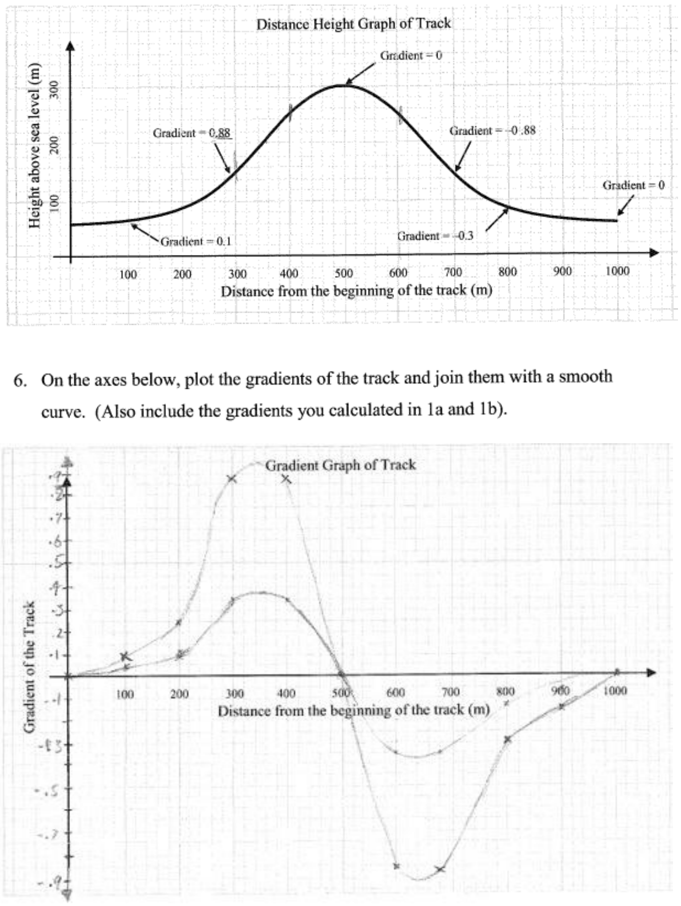
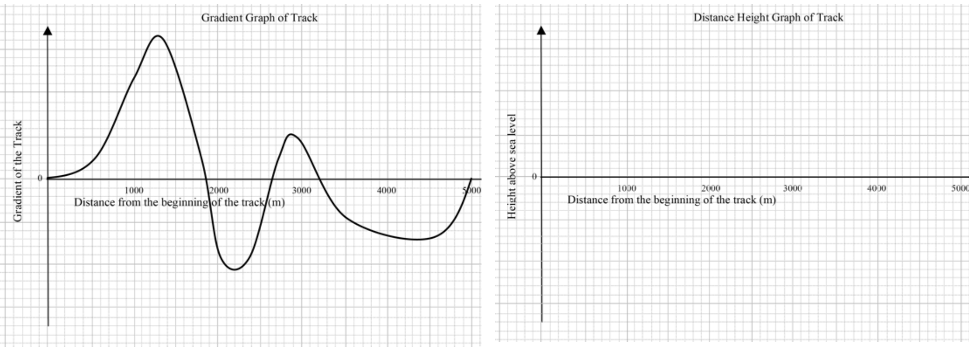


Figure 3. The graph of the initial tramping track and the gradient graph produced by Amy and Jay. Note, a second gradient graph was also drawn on the same axes for another tramping track of less amplitude.

After completing this task, Amy and Jay were then asked to work in the opposite direction. They were presented with a graph of the gradients of a tramping track (see figure 4) and asked to produce its distance-height graph. This was mathematically equivalent to finding the antiderivative of a function presented graphically, with no specific values given.



4a 4b

Figure 4. The tramping track task (the second graph, 4b, was presented directly below the first).

Amy suggested a method for numerically approximating the area, saying, "You could add up all the little squares and consider them to be one hundred…if you just do it that way adding up the squares". This corresponds to the Recognition phase of the RBC+C framework, since she recognised the potential value of a previous construct. However, attempts were quickly abandoned and she suggested a completely different approach.

Amy 421	Basically because this [*points at a function graph from figure 3*] is like kind of similar to that [*the gradient graph in figure 4a*] except this part [*points at the first minimum of graph 4a*] is smaller which says that… Which seems to mean somehow that this gradient doesn't decrease at the same rate it has increased here, so I'm thinking it might go up like this, and then this side coming down is not quite as steep [*traces pen along the function graph in the warm-up task*] because it doesn't continue, like this is not as big as here, and that's maximum steepness [*points at the gradient graph in figure 4a*]. It's steeper than the bit coming down [*traces pen along the first positive, decreasing section of the gradient graph*], …and then not quite as steep as and then a really flat gradient again and then go down again. Because there is another point where the gradient is zero, but it's not [*draws a rough sketch–see figure 5*].	Pattern matching approach Global and pointwise perspectives Interval perspective

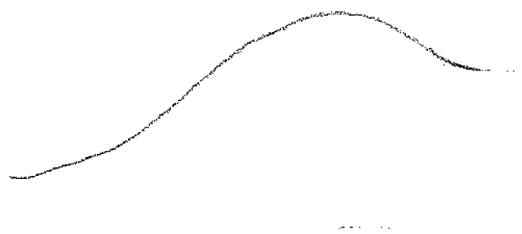


Figure 5. Amy's rough sketch of the first part of the antiderivative.

Speaker	Line #	Transcription and comments	Analysis
Amy	423	…I like this one [*points at her rough sketch in figure 5*]. How did I do it? I don't know if that makes sense, I'm not sure about this bit here basically [*circles the section in figure 4a where the graph is negative and then positive again– see figure 6*].	
Jay	424	That is true that when these are zero [*points at the second and third x intercepts in figure 4a*] then that should be..[*pause*]	Pointwise perspective; recognises the relevance of a zero gradient
Amy	425	So there should be two flat bits here [*points at figure 3*].	Builds-with the zero gradient to get a horizontal tangent on the antiderivative
Jay	426	One there, one there and one there, [*points at the second, third and fourth x-intercepts in figure 4a*] so there are three during the track itself.	Pointwise perspective; agrees with Amy about the zero gradients
Amy	427	Flat bits, and then decreasing [*looks at figure 4*]. Yeah there's got to be two other flat bits. Three other flat bits. Oh no wait a second because when we did this there was [*points at figure 4, and then looks confused*].	Starts the move to an interval perspective
	428	..[*pause*]	

Jay	429	This thing is this is just not going to work for every single..[*pause*]	
Amy	430	I reckon it's like this, it's a big hill and then a little hill [*traces this on her page*]	The breakthrough using an interval perspective
Jay	431	And then	
Amy	432	Because if you split this into two [*lays a pencil vertically through the third x-intercept in figure 4a*] you've got two graphs that are like this [*points at function graph at the top of figure 3*] except that some of the steepnesses are going to be different. Generally I think that..	Interval perspective; a pattern matching strategy is emerging; Amy has shifted her attention from the interval that was problematic for her (utterance 423, figure 6) by segmenting the given graph (figure 4a) into two intervals, each containing one maximum and one minimum
Jay	433	So, you're quite right.. then in this first half..[*pause*; *agrees*]	
Amy	434	A bit one and then this second on is like not as flat, flatter, like there is a steep hill and then a flat hill. [*traces this on the page*] You know how this graph went that way [*picks up one of the worksheets*] So we thought that that would be smaller so if you think about that and these ones are smaller than this then the second hill would be there. [*indicates this by pointing at her sketch of the antiderivative graph*].	Global interval perspective; recognises the value of the derivative approach; pattern matching using reverse thinking from the derivative graph in figure 3

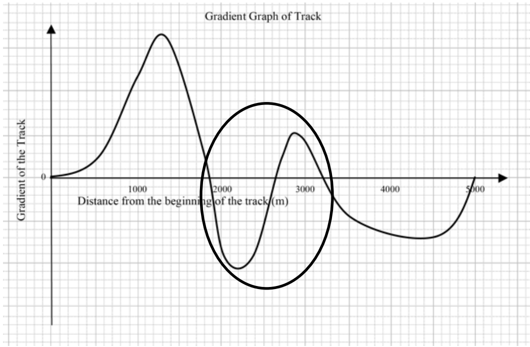


Figure 6. The parts of the graph circled by Amy, shown here in an oval (see Line 423)

Amy's identification of this pattern suggests the students are using the construct, 'similar graphs or sub-graphs have similar antiderivatives'. We see this in Amy's insight (see line 432) that the given derivative graph can be split into two parts, each interval with a similar shape but first "a big hill and then a little hill" (430), and hence a similar antiderivative. In turn, she notices that they can apply the warm-up activity (figure 3) in reverse to each of these interval parts to obtain two parts to the antiderivative, each having the general shape of the graph at the top of figure 3.

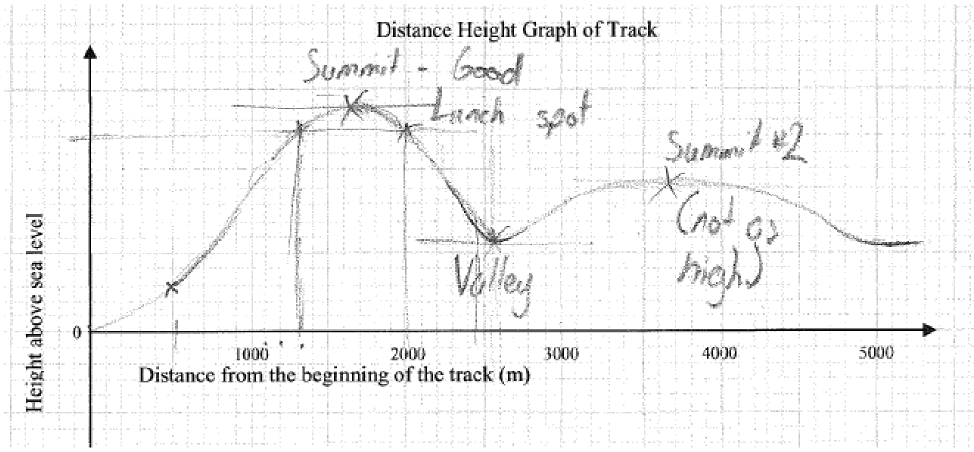


Figure 7. Amy and Jay's solution to the task using a pattern matching approach.

Her conjecture is that the difference between the two 'hills' or graph parts is that there would be "a steep hill and then a flat hill" (434). It is unclear how she arrived at the smaller 'hill', whether through the lower gradient for the second part or simply a wrong matching of the corresponding height of the relative maxima on the gradient graph. However, in this manner they were able to employ the similarity of shape construct and building-with it managed to complete the task to draw the antiderivative, as seen in figure 7.

4.2. Snapshot Two: Employing an interval perspective to construct turning points

In the second session[3], covered in this snapshot, we present evidence that Amy and Jay are able to reason with function intervals on either side of a zero value in the derivative function and to see the value of this perspective in enabling them to construct a maximum or minimum on the antiderivative graph. In doing so, Amy and Jay showed that they had constructed the following constructs, labelled E1-E4, which require both pointwise and interval perspectives:

E1: When the function is positive/negative at a given point the antiderivative is increasing/decreasing at that point. [pointwise perspective]

E2: When the function is positive/negative in a given interval the antiderivative is increasing/decreasing in that interval. [interval perspective]

E3: When the function is zero at a point the antiderivative has a turning point there. [pointwise perspective]

E4: When the function goes from negative to zero to positive on an interval then the antiderivative has a local minimum point there. [interval perspective]

Figures 8 and 9 show examples of Amy and Jay's reasoning during this session in support of the above claims. For example, in figure 8a we see a statement providing evidence that they had constructed E1 for the point $x = a$ on the given graph. Similarly, construction of E2 is demonstrated in figure 8b (where the interval is clearly marked) and E3 and E4 in figure 8c, where a local minimum at the point where $x = b$ is described along with reasoning via an interval perspective, that "the gradients [plural] are going from negative to positive".

[3] Transcript line numbering started from 1 again in this session.

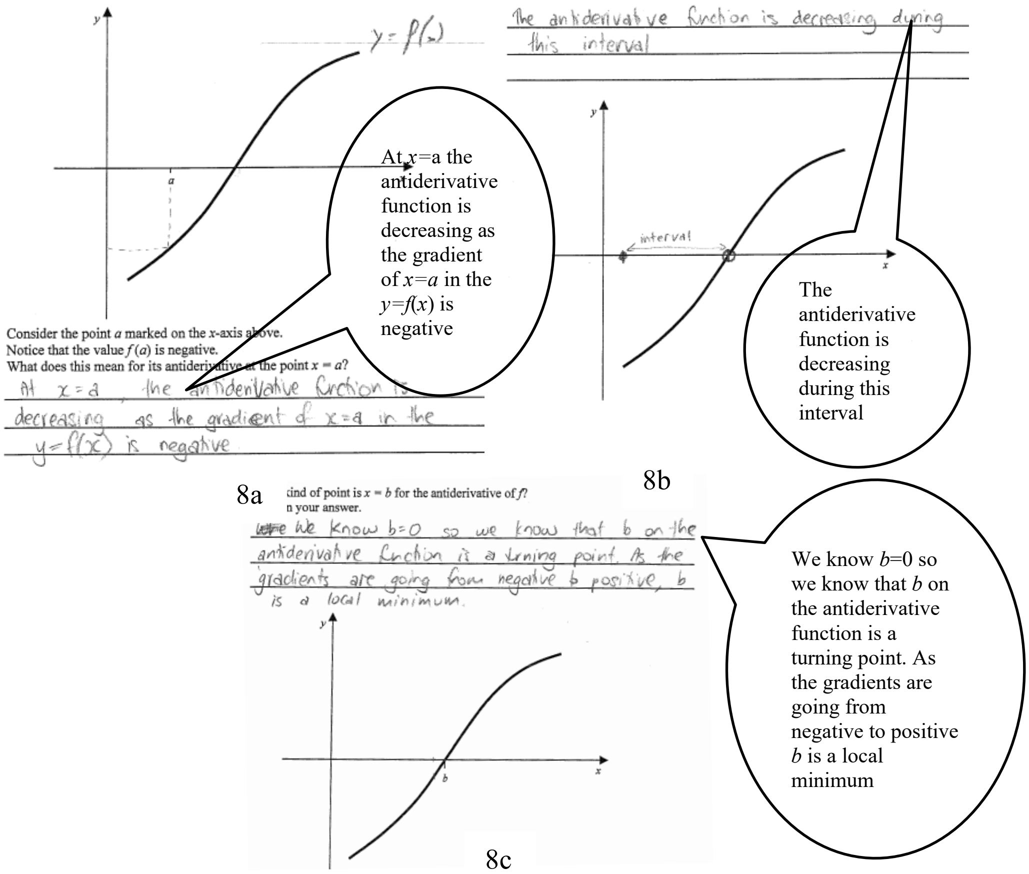


Figure 8. Evidence for construction of antiderivative ideas E1-E4.

They were also able to integrate their knowledge of these constructs in a building-with phase, correctly drawing the antiderivative graph of a decreasing function (see figure 9). In this process they state that the "gradients are going from positive to negative on the gradient function graph" and have constructed E5, closely related to E4, but identifying a local maximum. Once again they are clearly seeing the 'gradients' in an interval on either side of the zero value.

E5: When the function goes from positive to zero to negative on an interval then the antiderivative has a local maximum point there. [interval perspective]

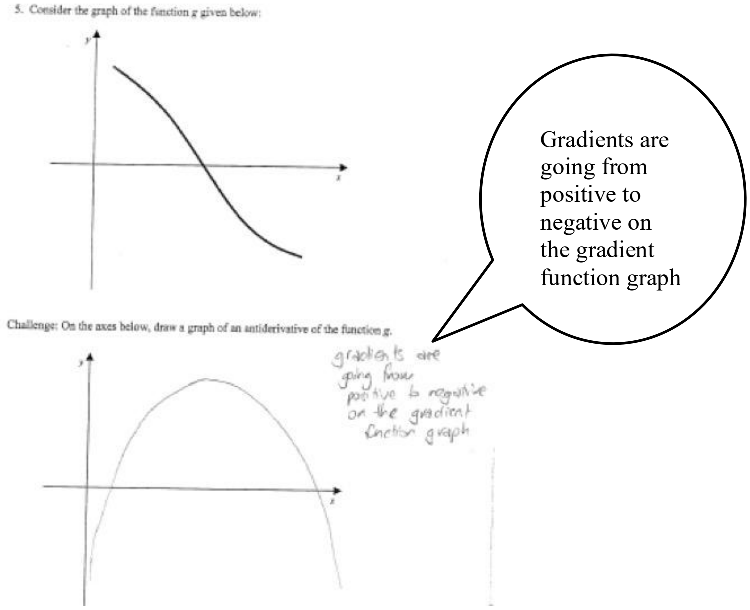


Figure 9. Using an interval perspective when building-with E1 – E4 to construct E5 and draw an antiderivative graph.

4.3. Snapshot Three: An interval perspective enables reasoning on a local maximum

Next, Amy and Jay work on questions designed to support them in the antiderivative of the function shown in figure 10. This raised the issue of how to deal with a maximum gradient. This snapshot details observations of Amy and Jay's reasoning on the intervals either side of a local maximum on the derivative function to construct information on the antiderivative graph.

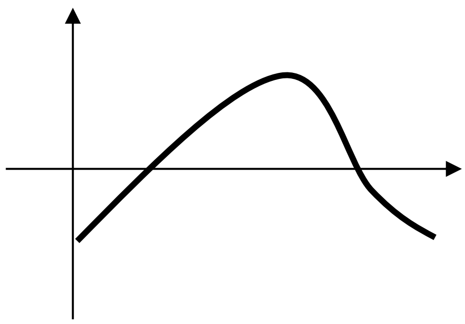


Figure 10. Amy and Jay are asked to draw a graph of the antiderivative of this function.

First, they recognise the usefulness of constructs E1-E5 above, and work with them, as they had previously done.

Jay	209	Yup, but this one starts as negative [*points at the positive, increasing part of the graph in figure 10*].	Recognises E2, on intervals
Amy	210	Negative to positive to negative again.	Interval perspective, confirms E2 relevant
Jay	211	So, turning points are here [*points at the two x intercepts in figure 10*].	Recognises and builds-with E3, at points where the function is zero—pointwise perspective
	…		
Amy	220	And then negative. That's kind of what we said eh? That means it's going to be going down, decreasing, and it's going to be..[*marks the interval between the second x-intercept and the right end point in figure 11*]	Interval perspective; builds-with E2
Jay	221	So basically like the two graphs we just did together.. like that [*draws a vertical line through the local maximum of the graph – see figure 11*].	Interval perspective; Dividing the graph into two intervals, he sees the similarity with the pattern of the two previous examples, and this enables them to build-with E4 and E5
Amy	222	So it's kind of like what we had eh?	Confirms the similarity

Here Jay uses an interval perspective to recognise (Line 221) that if the graph of the function in figure 10 is divided through its local maximum point then, using the two resulting intervals, it may be seen as a composite of the two types of graphs they had previously considered in figures 8 and 9. They then build-with this idea by inferring that they can combine their previous antiderivatives from these two tasks to draw the antiderivative of the graph in figure 10. In their solution the vertical line dividing the graph in two is clearly visible, as is the solution, which builds-with E4 and E5, combining a local minimum next to a local maximum. These are seen in their solution in figure 11. We can also see from the shaded areas in the first diagram along with the words 'negative', 'positive' and 'negative' written along the axis that they have used the interval perspective seen in E1 and E2 either as a check, or to help construct the antiderivative graphs.

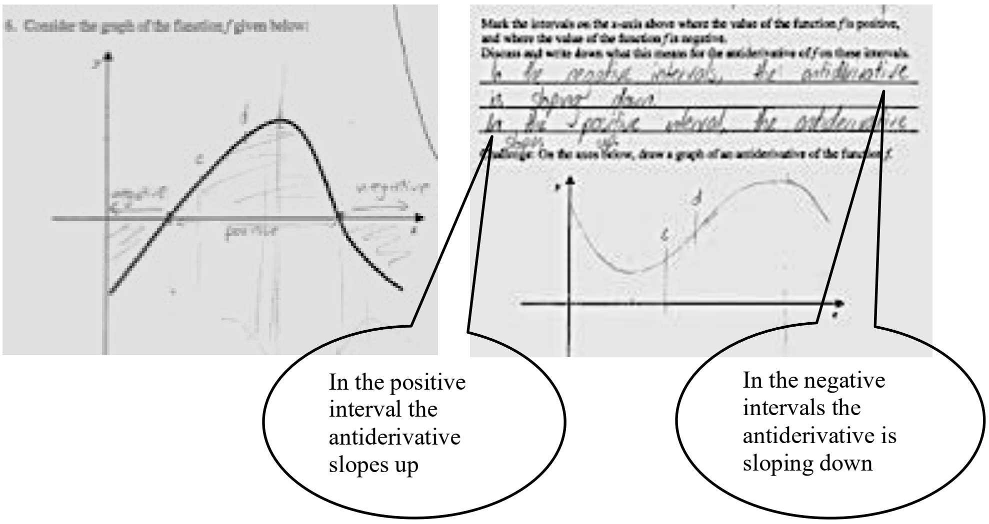


Figure 11. Amy and Jay's working and solution to the graph of the antiderivative of the function shown in figure 10. Note, points c and d were not present while Amy and Jay initially worked on the question but were added by them later (see 4.4).

However, while their pattern matching approach has enabled them to solve the problem it has not advanced to the point where they could construct a justification for why a positive/negative function corresponds to an increasing/decreasing antiderivative, and in particular they have not yet built the pointwise relationship between the local maximum of the function and the point of inflection on the antiderivative. In terms of the RBC+C framework we can say that they have recognised the relevance of constructs E1-E5 to the task, and built-with them to solve the problem, but have not produced a new construct.

4.4. Snapshot Four: Emergence of a new construct—Using an interval perspective to construct an inflection point for the antiderivative

In this final snapshot we briefly present evidence of how Amy and Jay reasoned from an interval perspective to build the construct of the antiderivative's point of inflection. The task instructed them to attend to two points, *c* and *d* on the graph (see figure 12), with the instruction: 'Consider the two points on the *x*-axis marked *c* and *d* below. At which point is the value of the function *f* greater? (i.e., is $f(c)>f(d)$ or vice versa?). Discuss and write down what this means for the antiderivative of *f* at $x = c$ and $x = d$.'

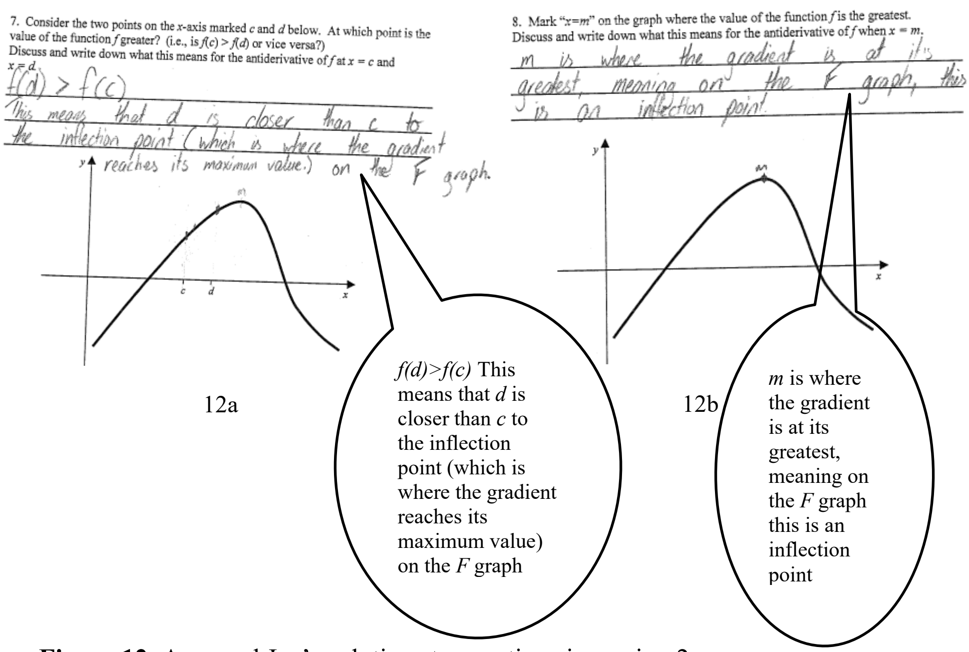


Figure 12. Amy and Jay's solutions to questions in session 2.

As Amy and Jay's discussed this question, they begin constructing a new and important idea about points of inflection. They have the versatility to consider an idea abstracted during school work on differentiation, a previous construct (PE).

PE: The maximum gradient of an increasing function, which changes from concave up to concave down through the point, occurs at a point of inflection.

We note that this construct, PE, requires both a pointwise perspective, considering the point of inflection, and an interval perspective, to reflect on the gradient of the function in an interval either side of the point, where it is either concave up or concave down.

Initially, they seem to be referring to the point at $x=c$ as a point of inflection (this point is approximately 'half way' along the interval where $y > 0$, and is where the gradient appears to be a maximum on the derivative graph), even though the function appears concave down throughout the interval containing c. This line of reasoning persists for a few minutes; later they focus on the maximum of the given function f.

Amy	282	It's going from, it's going from a positive grad, well it's either going from a positive gradient to a negative gradient, or a negative to a positive.	Interval perspective ; change in sign of gradient
Jay	283	Not necessarily, just basically at that point, [*points at the point of inflection on*	Pointwise perspective, PE: Maximum gradient at

		their antiderivative graph in figure 11] right where the inflection point is, [*points at the local maximum of f*] that is a maximum gradient there. So if you could draw a tangent line at that point [*sketches a little tangent line from the point of inflection on the antiderivative graph*] that would be the steeper, than anything on either side of it. The gradient decreases whichever way you go from that inflection point.	point of inflection. Interval perspective; the gradient decreases on either side of $x=m$– see figure 12
Amy	284	So this particular one we could say is a maximum [*points at the local maximum of f*]	pointwise perspective
Jay	285	Well that's true.	
Amy	286	It's going from positive, it's going from positive [*holds her pencil above the antiderivative graph and turns it, indicating the different tangent lines*].	Interval perspective using an embodied approach
Jay	287	It's still positive to here though. [*points at the second x intercept of f*] This is the least positive [*points at the first positive, decreasing section of f*].	Pointwise and interval perspectives
Amy	288	It starts, it goes from being positive, it starts to get.. [*indicates with the pencil the slope decreasing*]	Interval perspective, again embodied
Jay	289	To turn around, yeah, it gets there [*points at the second x intercept of f*] which is another maximum or minimum. It's a maximum, then.	Pointwise perspective; uses E5

As Amy and Jay reflect on the question "At which point is the value of the function *f* greater? (i.e., is $f(c)>f(d)$ or vice versa?)" (see figure 12), they connect the local maximum of *f* to the inflection point on the antiderivative graph.

Jay	306	Ah, OK, so what that means is, *m* is the inflection [*marks the local maximum on the function as m, see figure 12*]	
Amy	307	It's reaching that inflection point eh? [*points at the local maximum of f*] Like this is [*points at $f(d)$*].	Pointwise comparison

Jay	308	So d is steeper than c [*points at f(d) and then at f(c)*] Um. And it probably does lead to d is higher.. up.. on the derivative graph than c is.	The gradient at $x = d$ on the antiderivative is greater than the gradient at $x = c$
Amy	309	Because if it was down here [*points at the first negative, decreasing section of f*] then it would be the downward slope. Half way down the downward slope.	Interval perspective; uses E2
...	...		
Amy	313	Yeah, so..inflection point. So, um.. So, shall we write *f* of *d* [*f*(*d*)] is…	Pointwise perspective
Jay	314	Yeah let's do that.. [*Starts writing down the answer to question in figure 12a*] Um, which means.. that *d*.. is closer to the maximum or minimum, is that how you say it or..?	Still building the idea that the maximum point gives a maximum gradient

In lines 313 and 314 we see that Jay and Amy are using the idea of a maximum gradient at the point marked m to build the idea of the increasing gradient on an interval as points get closer to m. Thus a pointwise approach generalises to interval thinking through their idea of being, in general, 'closer to' m, i.e. in an interval. This idea of 'getting closer to' a point may indicate the start of the formation of a local perspective that becomes so important in the transition to tertiary mathematics. Their written solution shows this and describes the relationship between the gradient of *F*[4] corresponding to *f*(*c*) and *f*(*d*), saying that "*d* is closer than *c* to the inflection point (which is where the gradient reaches its maximum value) on the *F*", in figure 12a. So although they simply state that $f(d) > f(c)$ they are using the relative, pointwise comparison idea that being closer to the maximum gradient m implies a larger gradient, rather than using the general, notion that points 'higher up' on the graph (ie with a greater *y*-value) represent a steeper gradient. Hence, they have produced a new construct, E6, that identifies the location of a point of inflection on the antiderivative graph. In addition to constructing E6, there is evidence that they have consolidated it. In their discussion we see repeated, flexible recognition of this new construct. However, we have no direct evidence they are using an interval perspective to generalise to E7, that points closer to the maximum on the derivative graph represent a steeper gradient on the antiderivative graph.

---

[4] We use their notation *F* for the antiderivative of *f*.

E6: A local maximum on the function graph corresponds to a point of inflection with maximum gradient on the antiderivative graph.

E7: The greater the value of $f(p)$ on the function graph the greater the gradient at $x = p$ on the antiderivative graph F.

Following their discussion they wrote, as seen in figure 12b that "m is where the gradient is at its greatest, meaning on the F [antiderivative] graph, this is an inflection point." While this was written second, as an explicit answer to the question asked in figure 12b, it appears that this had been constructed while working on a later question.

In this snapshot we see that Amy and Jay were able to form links between school and university constructs, in this case between differentiation and antiderivative, which is an important aspect of student thinking during transition. Here, reflecting on the construct PE led them adopt an interval perspective to think about the gradient of the function in an interval either side of a point.

5. Discussion

The four snapshots presented above show that, given suitably designed activities, a graphical approach to antiderivative, such as the one described here, has the potential to engage students with thinking about functions in a pointwise, interval and global manner while they engage in constructing abstract knowledge. This was true of Amy and Jay, who employed each of these modes of thinking about function in their activity. Hence, we know that the graphical approach is accessible, at least for these students, who were of average ability, and takes the emphasis away from the standard algebraic manipulation that is often the norm. It would be useful to investigate whether the same results occur for a wider range of students. Moreover, it is conducive to the interval perspective, since in order to draw the antiderivative it was necessary for them to engage with functions on intervals. For example, in Snapshot 1 we saw how they used an interval perspective to compare sections of graphs and thus enable a pattern matching strategy to emerge. In the second snapshot Amy and Jay identified intervals where a derivative function is positive or negative so that the corresponding antiderivative is increasing or decreasing in that interval. They then combined these to form the construct that when a function goes from negative to zero to positive, or vice versa, on an interval then the antiderivative has a local minimum or maximum point. In Snapshot 3 we see how they were able to divide the domain of a derivative function into four intervals and apply the construct above to each pair in turn, producing an antiderivative with two turning points.

Constructing the point of inflection, seen in Snapshot 4, proved more challenging, but by reasoning from an interval perspective they were able to connect the local maximum of the derivative function to the inflection point on the antiderivative graph. Graphically constructing a point of inflection on an antiderivative graph is not straightforward, although Yoon et al. (2014) have described one way students may approach this, using gestures. The two students were able to recognise that there would be a point of inflection on the antiderivative graph, through the local maximum/minimum property of the derivative, and could build-with these constructs, using them to construct other ideas. However, while there was evidence of consolidating E6 (a local maximum on the function graph corresponds to a point of inflection) a later task (not presented here) required the construct that a minimum on the derivative graph implies a point of inflection on the antiderivative graph with minimum gradient. Interestingly, in this case they were unable to use it in new situations, so we conclude that it was not consolidated. This was partly because there was no opportunity for them to consolidate it prior to the final task and thus have it freely available. In addition, the final task required them to manage the difficult process of coordinating two constructs. Why was this construct that a local minimum on the function graph corresponds to a point of inflection with minimum gradient on the antiderivative graph partially obscured? Possibly due to the emphasis on their two primary strategies, the interval construct that when the derivative is positive/negative then the antiderivative is increasing/decreasing on the given interval, and interval pattern matching. Their emphasis was confirmed when they were asked to write how to distinguish graphs of derivatives and antiderivatives. In their explanation they wrote:

> If a function is a derivative of another function then when it is above the *x* axis the function it is a derivative of will be increasing. When the derivative function is below the *x* axis, the function will be decreasing.
> If one function's maximum or minimum matches up with a 2nd function's crossing the *x* axis, then the second function is a derivative of the first.

The first two sentences are a clear statement of the interval construct E2, while the third is reverse reasoning based on the pointwise constructs E4 and E5. Hence, we can deduce that all three of these constructs were consolidated by the students. There was also some evidence in Snapshot 4 that Amy and Jay were beginning to lay the foundation of a local perspective on function, reasoning on smaller and smaller intervals. For example, we see from line 283 that they were able to talk about the gradient being less steep either side of a maximum and so 'The gradient decreases whichever way you go from that inflection point', with the implication that this would be true even on a very small interval. Further, considering points to the left of the local maximum, as seen in lines 308-314 above, we can conclude that they were reasoning from an interval perspective, considering points 'higher up' on the graph and 'closer to the maximum or minimum', once again implying a small interval size.

In the light of the above we recommend stressing an interval perspective of function during the transition from school to university as a potential bridge to the local perspective needed at the tertiary level. It has the potential to assist with student understanding of the Fundamental Theorem of Calculus since it can assist in explaining why antiderivatives may be used during integration, something that many students in transition fail to appreciate. Another advantage for the secondary-tertiary transition of the kind of graph-based activity introduced here is that the tasks are suitable for secondary or tertiary students (such as those in our study) who have not been exposed to an interval perspective on function. This gives it value at either end of the secondary-tertiary transition.

Moving students away from a focus on a pointwise or global perspective of functions to include an interval one has been described as important in students' mathematical development toward a local perspective (Vandebrouck, 2011). In addition, the importance of including a graphical representation of function to help students in transition has been emphasised by Vandebrouck and Leidwanger (2016), who maintain that graphical tasks are important to assist understanding of limits. In the research presented here, there was little evidence of local thinking about functions since the graphical tasks did not require such an approach. However, the interval perspective is much closer to a local one than either a point or a global perspective, and hence it may be assumed that it is a useful starting point for progression to a local perspective. Confirming this assertion would be a useful subject for further research. Using the three perspectives on function the students were able to construct and consolidate a number of relevant mathematical constructs that they had not previously been exposed to. Although the construct that a local minimum on the function graph corresponds to a point of inflection with minimum gradient on the antiderivative graph had been constructed by Amy and Jay, their failure to consolidate it confirms the challenging nature of what is often an ongoing process, involving the need to fold back on previous ideas, and which may require several subsequent activities to do so (Hershkowitz et al., 2020).

An important aspect of the process of abstraction is the social nature of the building of new knowledge structures. It is unlikely that Amy and Jay would have made similar progress in abstraction if they had worked individually. Rather, being part of a small co-learning group (Jaworski, 2001) provides a "supportive community through which knowledge can develop and be evaluated critically" (Jaworski, 2003, p. 252). This support, encouragement and positive critique were seen throughout Amy and Jay's activity in this research. The implications for transition are that it is helpful for students both at school and university to learn new constructs in small groups. If students have this experience at school then continuing it in their tertiary experience can be beneficial. This may be achieved through tutorials but it has also been used in large lectures through the medium of flipped lectures, Problem-Based Learning (PBL), or a similar approach. Following an in-depth review of flipped lectures, Lo, Hew and Chen (2017) provide a suggested list of design principles for them that includes: Facilitate peer-assisted learning through small-group learning activities.

On several occasions we saw that Amy and Jay employed embodied, enactive thinking to trace curves. Such thinking with gestures has been shown to be beneficial in helping student construct formation (Yoon, et al., 2010, 2011) and the graphical environment seems to encourage this. Further, there was no evidence that Amy and Jay attempted tasks by resorting to algebraic methods. In their research Hong and Thomas (2014) found that a significant number of students try to solve graphical antiderivative problems by employing steps such as modelling the given graph algebraically, integrating symbolically and then sketching the resulting function. The reason this did not arise here may be that we considered the potential for this approach and so constructed graphs that did not resemble well known functions.

While it may not be necessary to make the transition from school to university mathematics fully smooth, activities that promote an interval perspective on function, which is central to many areas of mathematics at university, can only be of benefit to student learning and assist them to make the change. It has been shown that an emphasis on a pointwise and symbolic algebraic thinking in schools tends to produce students with a reliance on this form of working (Gray & Thomas, 2001). In many countries this kind of algebraic procedural work dominates school mathematics, but the use of graphs with no explicit algebraic function requires students to think in a qualitatively different manner that will no doubt assist in the transition to tertiary mathematics.

Bibliography

ARTIGUE, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematics Learning, 7*, 245–274.

CLARK, M. & LOVRIC, M. (2009). Understanding secondary–tertiary transition in mathematics. *International Journal of Mathematical Education in Science and Technology*, *40*(6), 755-776. doi:10.1080/00207390902912878

DE VLEESCHOUWER, M. (2010). An institutional point of view of the secondary–university transition: The case of duality. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 155-171. doi:10.1080/00207390903372445

DIAS, M., ARTIGUE, M., JAHN A., & CAMPOS, T. (2008). A comparative study of the secondary-tertiary transition. In M. F. Pinto & T. F. Kawasaki (Eds.). *Proceedings* *of the 34th Conference of the PME* (Vol. 2, pp. 129-136). Belo Horizonte, Brazil: IGPME.

DREYFUS, T., HERSHKOWITZ, R., & SCHWARZ, B. B. (2001). Abstraction in context: the case of peer interaction. *Cognitive Science Quarterly, 1,* 307-358.

DREYFUS, T., HERSHKOWITZ, R., & SCHWARZ, B. (2015). The nested epistemic actions model for abstraction in context – theory as methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping, N. Presmeg (Eds.) *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp 185–217). Springer, Dordrecht.

DREYFUS, T., & TSAMIR, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*(3), 271-300. https://doi.org/10.1016/j.jmathb.2004.06.002

EISENBERG, T., ENGELBRECHT, J., & MAMONA-DOWNS, J. (2010). Advanced mathematical topics: Transitions, evolutions and changes of foci. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 139-141. doi:10.1080/00207391003607870

ENGELBRECHT, J. (2010). Adding structure to the transition process to advanced mathematical activity. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 143-154. doi:10.1080/00207390903391890

FARMAKI, V., & PASCHOS, T. (2007). The interaction between intuitive and formal mathematical thinking: A case study. *International Journal of Mathematical Education in Science and Technology*, *38*(3), 353-365. doi:10.1080/00207390601035302

FERNÁNDEZ-PLAZA, J. A., RICO, L. & RUIZ-HIDALGO, J. F. (2013). Concept of finite limit of a function at a point: meanings and specific terms. *International Journal of Mathematical Education in Science & Technology, 44*(5), 699-710.

GHEDAMSI, I. (2008). *Enseignement du début de l'analyse réelle à l'entrée à l'université: Articuler contrôles pragmatique et formel dans des situations à dimension a-didactique.* Unpublished PhD Thesis. Université Bordeaux 2, France & Université de Tunis, Tunisie.

GONZÁLEZ-MARTÍN, A. S., NARDI, E., & BIZA, I. (2011). Conceptually-driven and visually-rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematical Education in Science and Technology, 42*(5), 565-589.

GRAY, R., & THOMAS, M. O. J. (2001). Procedural and conceptual interactions with quadratic equation representations. *Science and Mathematics Education Papers* (SAMEpapers), Waikato University, 113–128.

GUEUDET, G., & THOMAS, M. O. J. (2020). Secondary-tertiary transition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 762–766), Dordrecht: Springer.

HERSHKOWITZ, R., DREYFUS, T., & SCHWARZ, B. B. (2020). Abstraction in context. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_100032

HERSHKOWITZ, R., HADAS, N., DREYFUS, T., & SCHWARZ, B. B. (2007). Processes of abstraction from the diversity of individuals' constructing of knowledge to a group's "shared knowledge". *Mathematics Education Research Journal, 19*(2), 41–68. https://doi.org/10.1007/BF03217455

HERSHKOWITZ, R., SCHWARZ, B. B., & DREYFUS, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*(2), 195-222. https://doi.org/10.2307/749673

HONG, Y. Y., KERR, S., KLYMCHUK, S., MCHARDY, J., MURPHY, P., SPENCER, S., THOMAS, M. O. J., & WATSON, P. (2009). A comparison of teacher and lecturer perspectives on the transition from secondary to tertiary mathematics education. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 877-889.

HONG, Y. Y., & THOMAS, M. O. J. (2014). Graphical construction of a local perspective on differentiation and integration. *Mathematics Education Research Journal*. *27,* 183-200. DOI: 10.1007/s13394-014-0135-6

JAWORSKI, B. (2001). Developing mathematics teaching: Teachers, teacher-educators and researchers as co-learners. In F.-L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 295–320), Dordrecht: Kluwer.

JAWORSKI, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics, 54*, 249–282.

JENNINGS, M. (2011). The transition from high school to university: The university of Queensland perspective. In J. Hannah, J. & M. O. J. Thomas, (Eds.) *Proceedings of Volcanic Delta, The Eighth Southern Hemisphere Conference on Teaching and Learning Undergraduate Mathematics and Statistics* (pp. 139-149), Rotorua, NZ: The University of Canterbury and The University of Auckland.

JONES, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *Journal of Mathematical Behavior, 38*, 9–28

LEVIATAN, T. (2008). Bridging a cultural gap. *Mathematics Education Research Journal*, *20*(2), 105-116.

LO, C. K., HEW, K. F., & CHEN, G. (2017). Toward a set of design principles for mathematics flipped classrooms: A synthesis of research in mathematics education, *Educational Research Review, 22*, 50-73.

MAMONA-DOWNS, J. (2010). On introducing a set perspective in the learning of limits of real sequences. *International Journal of Mathematical Education in Science and Technology, 41*(2), 277-291.

MARTINO, P. D., GREGORIO, F., & IANNONE, P. (2023). The transition from school to university in mathematics education research: New trends and ideas from a systematic literature review. *Educational Studies in Mathematics, 113*, 7-34. https://doi.org/10.1007/s10649-022-10194-w

MONTOYA-DELGADILLO, E., PÁEZ MURILLO, R-E., VANDEBROUCK F., VIVIER L. (2018). Deconstruction with localization perspective in the learning of analysis, *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 139–160. https://doi.org/10.1007/s40753-017-0068-z

OEHRTMAN, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, *40*(4), 396-426.

SCHWARZ, B. B., DREYFUS, T., & HERSHKOWITZ, R. (2009). The nested epistemic actions model for abstraction in context. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11-41), Abingdon: Routledge.

THOMAS, M. O. J. (2008). Developing versatility in mathematical thinking. *Mediterranean Journal for Research in Mathematics Education, 7*(2), 67-87.

THOMAS, M. O. J. (2023). Mind the gap: Reflections on collaboration in research and teaching. In S. Stewart (Ed.) *Mathematicians' reflections on teaching: A symbiosis with mathematics education theories* (pp. 53-72). Springer.

THOMAS, M. O. J., DE FREITAS DRUCK, I., HUILLET, D., JU, M-K., NARDI, E., RASMUSSEN, C., & XIE, J. (2015). Key mathematical concepts in the transition from secondary school to university. *Proceedings of The 12th International Congress On Mathematical Education (ICME-12) Survey Team 4* (pp. 265-284). Seoul, Korea.

THOMAS, M. O. J., HONG, Y. Y., & OATES, G. (2017). Innovative uses of digital technology in undergraduate mathematics. In E. Faggiano, A. Montone, & F. Ferrara (Eds.) *Innovation and technology enhancing mathematics education* (pp. 109-136). New York: Springer.

THOMAS, M. O. J., & KLYMCHUK, S. (2012). The school-tertiary interface in mathematics: Teaching style and assessment practice. *Mathematics Education Research Journal, 24*(3), 283-300.

THOMPSON, P. W., & CARLSON, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

THOMPSON, P. W., & SILVERMAN, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America. Available at http://pat-thompson.net/PDFversions/2008MAA Accum.pdf

TREFFERS, A., & GOFFREE, F. (1985). Rational analysis of realistic mathematics education. In L. Streefland (Ed.), *Proceedings of the 9th International Conference for the Psychology of Mathematics Education* (Vol. II, pp. 97-123). Utrecht, The Netherlands: OW&OC.

VANDEBROUCK, F. (2011). Students' conceptions of functions at the transition between secondary school and university. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2093-2102). Rzeszow, Poland.

VANDEBROUCK, F., & LEIDWANGER, S. (2016). Students' vizualisation of functions from secondary to tertiary level. *The First conference of International Network for Didactic Research in University Mathematics*, Montpellier, France. hal-01337928

YOON, C., DREYFUS, T., & THOMAS, M. O. J. (2010). How high is the tramping track? Mathematising and applying in a calculus model-eliciting activity. *Mathematics Education Research Journal, 22*, 141-157

YOON, C., THOMAS, M. O. J., & DREYFUS, T. (2011). Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics, 78*(3), 371-393. DOI 10.1007/s10649-011-9329-y.

YOON, C., THOMAS, M. O. J., & DREYFUS, T. (2014). The role of conscious gesture mimicry in mathematical learning. In L. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 175-195). Charlotte, NC: Information Age Publishing.

MIKE THOMAS

Auckland University

moj.thomas@auckland.ac.nz

TOMMY DREYFUS
Tel Aviv University
tommyd@tauex.tau.ac.il

CAROLINE YOON
Auckland University
c.yoon@auckland.ac.nz