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CALCULUS EDUCATION: ASPECTS OF ORDER, CONTINUITY, AND RECONCEPTUALIZATION

Abstract. The triad order, continuity, and reconceptualization that appears in the title of this paper refers to a juxtaposition of three aspects of calculus education. Order refers to differentiation followed by integration (DI approach) versus integration followed by differentiation (ID approach) versus Thompson's integrated approach (TI approach)[1] which view differentiation and integration inseparable. Continuity refers to the impact of these approaches on student learning as they transition from high school to university. Reconceptualization refers to the effort to reform calculus learning and teaching by reeducating future secondary teachers relearn calculus concepts and ideas through the lens of quantitative reasoning. This is an analytic paper. It begins with an analysis of the cognitive and pedagogical features of the three approaches, DI, ID, and TI, and continues with a discussion of the continuity problem concerning the transition from school mathematics to university mathematics, focusing on the difficulty to reform calculus education in the U.S. To advance this reform, it is necessary to examine in depth the current approaches to calculus education in the U.S., as well as alternative approaches advocated by mathematicians and mathematics education scholars. The analysis of the three approaches, DI, ID, and TI, aims at contributing to this essential examination. As part of this examination, the paper offers a calculus module for prospective secondary teachers who have already taken the "mainstream" calculus sequence. The module, while akin to the TI approach, its development and implementation rest on a separate theoretical framework.

Key words. Teacher Calculus, order, continuity, intellectual needs.

Résumé. La triade ordre, continuité et reconceptualisation, mentionnée dans le titre de cet article, fait référence à une mise en relation de trois dimensions de l'enseignement de l'analyse réelle**.** L'ordre renvoie à trois approches distinctes : la différentiation suivie de l'intégration (approche DI), l'intégration suivie de la différentiation (approche ID), et l'approche intégrée de Thompson (approche TI), qui considère la différentiation et l'intégration comme indissociables. La continuité concerne l'impact de ces différentes approches sur l'apprentissage des étudiants, notamment lors de la transition du lycée à l'université. La reconceptualisation désigne les efforts visant à réformer l'enseignement et l'apprentissage de l'analyse en invitant les futurs enseignants du secondaire à redécouvrir ces concepts à travers le prisme du raisonnement quantitatif. Cet article est de nature analytique. Il débute par une analyse des caractéristiques cognitives et pédagogiques des trois approches (DI, ID et TI), puis aborde la problématique de la continuité dans le passage des mathématiques scolaires aux mathématiques universitaires, en insistant sur les difficultés

[1] The integrated approach is attributed to Thompson even though others, including the author of this paper, hold a similar pedagogical view. Thompson's theoretical and empirical research on this matter is the most comprehensive to date.

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rencontrées dans la réforme de l'enseignement du calcul aux États-Unis. Pour faire progresser cette réforme, il est nécessaire d'examiner en profondeur les approches actuelles de l'enseignement du calcul aux États-Unis, ainsi que les alternatives proposées par les mathématiciens et les didacticiens des mathématiques. L'analyse des trois approches (DI, ID et TI) vise à contribuer à cet examen essentiel. Dans ce cadre, l'article propose également un module de l'analyse destiné aux futurs enseignants du secondaire ayant suivi le cursus classique. Ce module, bien qu'il s'inscrive dans une logique proche de l'approche TI, est conçu et mis en œuvre à partir d'un cadre théorique distinct.

Mots clés. Analyse réelle du professeur, ordre, continuité, besoins intellectuels.

Calculus is one of two courses—the other being linear algebra—required by all science, engineering, economics, and mathematics students, the number of whom is likely to be in the hundreds of thousands. Conceptual understanding and mastery of procedures of calculus are essential for success in science, technology, engineering, and mathematics (STEM) careers. Hence, improving calculus education is a key component of the effort to improve STEM education of students in their first two years in college (FCSETF, 2012). In line with this goal and to contribute to the collaborative work in my institution to improve the quality of the calculus sequence, I conducted a series of seminars on the learning and teaching of calculus. The participants were practitioners—schoolteachers and mathematicians who teach calculus—as well as mathematics education researchers. The discussions in the seminar revolved around publications on the learning and teaching of calculus, which the participants read in advance. The mix of practitioners and researchers yielded lively discussions, filled with insights and rich in questions concerning practical matters as well as theoretical issues. This paper focuses on three of the central issues raised in the seminar: order, continuity, and conceptualization.

Order. Order refers to sequencing of calculus content. This issue was inspired by two sources: The first source was Bressoud's (2019) thorough historical account of the development of calculus, which led him to conclude that "a strong case can be made for starting calculus with an introduction to integration via accumulation." (p. 187). The second source was the fact that while most calculus textbooks introduce first differentiation followed by integration (hereafter DI), there exist those which structured their content in the reverse order, integration followed by differentiation (hereafter ID). Parrott (2014) surveyed 116 calculus texts and found seven textbooks that introduce integration first, the most recent of which appeared in 1979. The question is, then, what are the rationales expressed by the authors of textbooks who adopt the uncommon ID approach? The most notable of these texts are the books by Apostol (1967) and Rodin (1970). Apostol's rationale for his approach is akin to Bressoud's historical claim:

> The approach in this book has been suggested by the historical and philosophical development of calculus and analytic geometry. For example, integration is treated

> before differentiation. Although to some this may seem unusual, it is historically correct and pedagogically sound. Moreover, it is the best way to make meaningful the true connection between the integral and derivative. (Apostol, 1967, p. vii.)

Rodin's rationale is pedagogical:

> Personally, I … came to favor the "integration first" approach because I know of no better way to arouse the students' enthusiasm for the subject than to show them, on the very first day of class, how to find the area under a parabola. In the succeeding weeks of the course further accomplishment, significant and new in the student' eyes, follows quickly. The students seem to develop a real feeling for the Rieman integral and its uses. They also develop a faith in the power of calculus which provides the motivation to carry them through subsequent topics which are more difficult and have less immediate uses. Another advantage of this approach is that it allows a gradual introduction to limits; thus, the limit of a sequence is the only limit notion used in Chapter 1. Finally, it should be mentioned that this approach does no harm to students who require an early grasp of calculus techniques for use in their science courses. Since the user of this book gets into calculus immediately, he develops the techniques for doing physics problems involving work and acceleration faster than with most other texts. (Rodin, 1970, Preface)

To the dichotomy of DI versus ID was added Thompson's research-based integrated approach (hereafter TI), which views rate of change and accumulation as inseparable (see Project DIRACC in http://patthompson.net/ThompsonCalc).

Continuity refers to the discontinuities in mathematical learning that students encounter as they transition from high school mathematics to university mathematics (Artigue, 1999). In this paper, the focus is on the transition from high school calculus to university calculus. An analysis of the continuity problem reveals that due to the unique character of calculus education in the U.S., reforms are difficult, notwithstanding the locally productive efforts by scholars such as Thompson (see, for example, Thompson and Ashbrook (2019), Ely (2021), Moreno-Armella (2021)). To advance this reform, it is necessary to examine in depth current approaches to calculus education in the U.S., as well as alternative approaches advocated by mathematicians and mathematics education scholars. The paper offers an analysis of cognitive and pedagogical aspects of the three approaches as part of the collective effort to contribute to this examination.

Reconceptualization refers to the effort to reform calculus learning and teaching by reeducating future secondary teachers relearn calculus concepts and ideas through the lens of a particular theoretical framework. The choice of the words "reeducating" and "relearn" rather than "educating" and "learn" is to stress that the goal is not to replace the "mainstream" calculus sequence that undergraduate students, including secondary mathematic majors, are required to take—that is unlikely to be feasible due to social conventions and institutional constraints; rather, the goal is to offer a follow-up module aimed at helping prospective teachers rectify the knowledge they acquired in the “mainstream” calculus courses through acquisition of new ways of understanding and ways of thinking.

As was mentioned earlier, the paper is analytical, not empirical. It raises three central questions in hope to generate further analyses, both analytical and empirical. The questions are: 1) Which should be taught first, differentiation or integration, or should the two be inseparable? 2) Does order impact student learning? 3) What is the relative impact of the three approaches on calculus learning as students transition from high-school mathematics to tertiary mathematics?

The first question attends to theoretical analyses—historical, philosophical, cognitive, and pedagogical—of the three approaches. The second question attends to empirical investigations on the relative efficacy of the three approaches. This question, however, can be examined in a range of populations and contexts (e.g., university students versus community college students versus high-school students; and mathematics majors versus science majors, versus engineering majors). Of these ranges of possibilities, the third question aims at a particular transition—the transition from high-school calculus to university calculus. Additional research questions entailed from the paper’s analyses are outlined in the concluding section.

The paper is structured around two parts. The first part is comprised of Sections 2 and 3. They address the order and continuity issues, as outlined above. The second part is comprised of Sections 4 and 5. They discuss elements of the module, including (a) an outline of the principles that guided the development and implementation of this module, central among which is the principle of intellectual need—local versus global; and (b) a sample of instructional activities illustrating the philosophy underlying the module, as well as an outline of the phases constituting the learning trajectory of the Fundamental Theorem of Integral Calculus. The paper concludes with a summary and further research questions that arise from the discussions of the previous two parts. As was stated earlier, this paper is theoretical, not empirical. As such, the episodes accompanying the discussions do not purport to serve as supporting empirical evidence; rather, they are merely illustrations for the analyses addressed in the paper.

1. The continuity problem in mathematics learning and its irrelevance to calculus education in the US

The problem of continuity in mathematics learning is neither new nor unique to the transition from high school mathematics to tertiary mathematics. It is a phenomenon reported in the literature, mostly in the context of elementary mathematics—what is known as domain shift (Nesher & Peled, 1986). For example, as students transition from the domain of whole numbers to the domain of rational numbers, they encounter difficulties resulting from their tendency to attribute properties of the original domain to the more general one. Or, in the case of the transition from whole numbers to integers, there is a discontinuity in the conceptualization of the concept of number, from a representation of capacity to a representation of a lack thereof. In general, researchers have recognized the fundamental conceptual change students must experience to successfully transition from elementary grade mathematics to middle-grades mathematics and from middle-grades mathematics to high-school mathematics; and, in Artigue's (1999) work, from high school mathematics to undergraduate mathematics.

Students' difficulties resulting from the lack of continuity can be situated in Brousseau's theory of didactical situations, specifically in his well-known distinction between epistemological obstacles and didactical obstacles (Brousseau,1997, Duroux, 1982). Some of the difficulties students experience through this transition are epistemological—unavoidable obstacles due to the intrinsic meaning of the concept; others didactical—avoidable obstacles due to narrow instruction or insufficient long-term curricular planning. Obstacles, didactical or epistemological, occur in any learning process, especially in the transition between domains, as was outlined earlier. Artigue illustrates such phenomena with compelling examples of obstacles students encounter as they transition from high-school calculus to elementary real analysis, since in France, her home country, analysis forms the curricular subject matter of undergraduate single variable calculus. The claim I put forth in this paper is that in the U.S., the continuity problem pertaining to the transition from high-school calculus to university calculus is not that there is a lack of continuity; on the contrary, the problem is that continuity is preserved through the transition. I explain.

Bressoud (2021) analyzed status studies dealing with calculus curriculum and instruction in the U.S. He points out that calculus education in the U.S. is impacted by two factors: lack of equity in public school funding and uniformity of the calculus sequence offered by universities.

> In the U.S., most public-school funding is raised within the community in which the schools are located. Because of this, decisions on funding levels, curricula, and staffing are made locally. This means that there is tremendous disparity across the country in what courses are offered and how teachers are prepared to teach these courses, [especially calculus]. Because of the lack of uniformity in access to calculus in high school and widespread reluctance by universities to recognize calculus learned in high school, most college [first calculus course] classes … contain students who are completely new to the terminology and concepts of calculus and students who have already demonstrated proficiency in all of the topics to be covered in that course. … [Furthermore,] in most U.S. universities, … there is usually one "mainstream" calculus sequence [whose emphasis is on] procedures and common applications … [The] high school [calculus] course … corresponds to what is taught in almost all colleges and universities. (Bressoud, 2021, pp. 522-3)

As to the quality of the cognitive demand, reflected in the rigor level of "mainstream" calculus courses, the finding from a comprehensive study by Tallman, Reed, Oehrtman, and Carlson (2021) draws a bleak portrait across a range of fundamental calculus concepts and ideas. Reforms efforts carried out in the last few decades do not seem to bring about a desired change. Thompson, Byerley, and Hatfield (2013) quote Tall (2010, p. 2) reflecting on 40 years of calculus reform: "After reform projects have attempted a range of different approaches using technology, what has occurred is largely a retention of traditional calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation." Thus, in the U.S. the course sequence offered by the university and their corresponding courses in the high school are largely identical, comprised mainly of procedures and graphical applications. Consequently, the cognitive demand of these courses is uniformly low, allowing a "smooth" transition throughout calculus education in high school and university. Efforts to reform calculus education in the U.S. have largely been unsuccessful.

To reform calculus education, it is necessary to examine in depth current approaches to calculus education in the U.S., as well as alternative approaches advocated by mathematicians and mathematics education scholars. The next section offers such an examination, narrowly focusing on the three approaches, DI, ID, and TI. The examination explores the features of the first two approaches and compares them to those of the TI approach. The assumption is that such an examination can contribute to a resolution of the continuity problem I have just outlined. This is so, because by better understanding the three approaches, researchers, curriculum developer, and practitioners will be better positioned to address the specific curricular and instructional elements needing reform and, consequently, be able to better differentiate between the levels of rigor required and feasible in high school calculus and those in university calculus.

2. Features of the DI-, ID-, and TI-Based Approaches to Calculus

As was mentioned earlier, most calculus textbooks introduce first differentiation followed by integration. Roughly speaking, the two varieties of textbooks contain mostly the same topics. ID-based textbooks start with accumulation through the topics, approximation of area, area as limit, the Riemann integral, and integration formulas, followed by rate of change through the topics, tangent line to a curve, derivative, derivative formulas, chain rule, and mean-value theorem. DI-based textbooks, on the other hand, start with the rate of change topics, followed by accumulation topics. In both cases, the FTIC is presented as the glue cementing the connection between accumulation and rate of change. Our search of the literature found no report on empirical studies comparing the impact of the two approaches on student learning. The TI-based approach uncovers a similar range of topics, while maintaining a strong conceptual connection between the ideas of rate of change and accumulation. This approach was developed through a sequence of teaching experiments and yielded significantly positive learning outcomes (Thompson, 2016).

On its face, the three approaches, DI versus ID versus TI, seem a matter of curriculum sequencing. As we will see below, the differences among the three approaches are not just a matter of order of curricular content but a matter of diverse views on cognition and instruction. In particular, the contrast between the ID and DI, on the one hand, and the TI on the other hand, is inherently philosophical. The discussions in the seminar, combined with claims expressed by teachers, university faculty, and textbook authors, yielded the following rationales.

Rationales in favor of the ID approach include:

ID1: Application power. The ID approach is the quickest way to tackle an interesting and an important mathematical problem—finding the area under a curve.

ID2: Continuity with past knowledge. The ID approach provides continuity to what students already know. By the time students enter college, they have gained an intuitive understanding of the concept of area and have learned how to calculate the area of various figures, such as rectangles, triangles, trapezoids, and circles. Building on this knowledge, the question of how to compute the area bound by a curve is natural and motivating, which may instill in the student a faith in the power of calculus. As was pointed out by Rodin (1970) in the preface to his book, "… I know of no better way to arouse the student's enthusiasm for the subject than to show them, on the very first day of class, how to find the area under a parabola".

ID3: Greater familiarity. The physical processes modeled by integration are simpler and more familiar than those modeled by differentiation. Because of this, integration is a simpler context in which to introduce the idea of a limit, a fundamental concept of calculus.

ID4: Lower cognitive demand. Furthermore, the concept of limit is introduced in and applied to sequences, not general functions. The notion of "limit of function" is more cognitively demanding than that of "limit of sequence".

ID5: Greater curricular space for conceptual understanding. The fact that the computation of integrals is less mechanical allows the focus to shift away from computation and toward conceptual understanding. More time could be spent modeling integration as the limit of Riemann sums, whose computation could be handed off to readily available software.

ID6: Consistent with historical development. The ID approach mirrors the historical development of calculus, in that two-thousand-year quest for calculating area and volume, beginning with Archimedes of Syracuse (circa 287-212 BCE), culminated in the discovery of a systematic method for these calculations by Newton and Leibniz in 17th Century.

The features of the DI-based approach are marked by the following claims made in its favor.

DI1: Formulation of a fundamental concept. The DI approach starts with a fundamental concept of calculus, rate of change, by asking: What is speed and how do we calculate it?

DI2: Simpler derivation of computational rules. Derivations of general formulas for derivatives are relatively simpler than the derivations of the general formulas for integration. Furthermore, the latter are not always applicable (often for simple-looking functions, such as $\int e^{x^2}\,dx$), whereby creating uncertainty for the students.

DI3: Simpler computations. Calculations of derivatives can be carried out mechanically based on a small number of simple rules. On the other hand, calculations of integrals are typically more complicated, often requiring ingenuity. Compare, for example, between the processes involved in differentiating and integrating the function $f(x) = \frac{x^2}{\sqrt{x^2-1}}$. While differentiation of this function involves merely a mechanical application of the Product Rule, Quotient Rule, and Chain Rule, integration requires a decision as to what integration technique to use, and if, for example, one decides to apply the Integration by Parts method, one needs to choose the parts of integration.

DI4: Historical neutrality. Calculus as a coherent theory did not begin with the study of accumulation problem, nor did it begin with the study of differentiation (as was pointed out by Bressoud (2019)). The reflexive role between differentiation and integration as reflected in the FTIC points to a "historical symmetry" between the two.

The TI-based approach features were extracted from one of the publications that outlined them (Thompson et al, P. 127).

TI1. The learning journey in this approach is "permeated with ideas of variation, covariation, and function as an invariant relationship between covarying quantities. In other words, it is a demand of the course that students think that variables vary—always."

TI2. The second feature is that "students must capture processes of variation, change, and accumulation symbolically—and their symbolizations must work, [in the sense] that when students create a function as a solution to a problem, …, the function must be both syntactically coherent … and … actually answers the question asked."

TI3. The third feature is that students' construction of meaning is central. Powerful meanings suggest courses of possible action in problematic situations to a person having them.

TI4. The fourth feature is that students accept open-form representations of a function, [e.g., $\int_0^x \cos(t)\,dt$ representing the exact accumulation of a quantity that changes at the rate of cos(t) within the interval $0 \le t \le x$], as actually representing the function [in the close form expression sin(x)].

A common characteristic to the ID and DI approaches is the typical view that calculus is comprised of two parts that exist independently but are tied together with the Fundamental Theorem of Integral Calculus (Thompson, et al. 2013). The differences between the two approaches might be cognitively and instructionally impactful in student learning. This is a question for empirical studies, whose results dependent on many curricular and instructional variables. However, theoretically, the two approaches seem to converge into one journey with a common destination—the Fundamental Theorem of Integral Calculus. The considerations stated seem to define a journey that advances through two paths which differ only in their starting points and the perceived didactical obstacles involved in each journey. The TI-based approach, on the other hand, is distinctively different in that its features—and hence its instructional goals—are formulated in terms of ways of thinking: TI1: thinking in terms of functions; TI2: referential symbolic representation and manipulation; TI3: inferences of meaning in the course of problem solving; and TI4: encapsulation of open form definition of functions.

This preliminary level comparison is insufficient to conclude relative impact on student learning. However, it may suggest that considerations of order alone are not the answer to the reform needed in calculus education. What is needed is deep overhaul of the curricular objectives and mode of instructional delivery through innovative initiatives. Thompson's DIRACC project (Thompson and Ashbrook, 2019) and Ely's (2021) differential-based approach are examples of promising approaches for the needed reform.

In Section 5, I will outline a project that tackled the "continuity problem" discussed earlier through a module designed for prospective secondary teachers. The module does not replace the "mainstream" calculus course but follows it. The goal of the module is to re-educate the prospective teachers through reflection and reconceptualization of some of what they have learned in their first "mainstream" calculus courses.

The design of the module and its instructional implementation were modeled through the principles of DNR-based instruction in mathematics (DNR, for short). DNR was discussed in length in various publications (e.g., Harel, 2008 a,b,c; Harel, 2013), including its implementation in the learning and teaching of multivariable calculus (Harel, 2021), and so it will not be elaborated upon here. However, to make the paper self-contained within its page limit, elements of DNR will be discussed briefly as needed. One of these elements is the notion of intellectual need.

3. Intellectual need: local versus global

One of DNR's (eight) premises is the knowledge-knowing-linkage premise. It states: any piece of knowledge humans know is an outcome of their resolution of a problematic situation they view as such (Harel, 2008a,b,c). Such situations do not occur haphazardly; rather, they emerge along complex paths paved by ever-changing social, cultural, and scientific needs. Historical evolutions of problems and their solutions coalesce into mathematical fields—calculus, algebra, geometry, etc.—yielding along the way new problems, which in turn necessitate new concepts and ideas often leading to the formation of new subfields (e.g., functional analysis, topology, number theory). The process continues through cycles of problems regulated by newly emerging societal and scientific needs. Through historical retrospective lenses, these scientific fields are organized around central overarching problems, referred to in this paper as the field's locus of global intellectual need (LoGIN, for short).

Intellectual need is a construct whose technical definition rests upon several DNR premises (see Harel, 2013). It is sufficient to define it here as a perturbational state an individual (or community) experiences as he or she encounters a situation that is incompatible with or presents a problem that is unsolvable by her or his current knowledge. Such a problematic situation, prior to its resolution, is referred to as an individual's intellectual need: It is the need to reach equilibrium by learning a new piece of knowledge. Thus, intellectual need has to do with disciplinary knowledge being created out of one's current knowledge through engagement in problematic situations conceived as such by the person. For example, many central concepts of real analysis—and some argue the entire field of real analysis (Bressoud, 2007)—were intellectually necessitated from Fourier's solution to Laplace's equation, $\frac{\partial^2 z}{\partial w^2}+\frac{\partial^2 z}{\partial x^2}=0$. In particular, Fourier's solution as an infinite cosine series led to a reconceptualization of the concept of function and included covariations not conceived as a function until then (e.g., the expansion $f(x)=\frac{\pi}{4}\left(\cos\frac{\pi x}{2}-\frac{1}{3}\cos\frac{3\pi x}{2}+\cdots\right)$. Intellectual need is different from *psychological need*, which has to do with people's desire, volition, interest, self-determination, and the like. Indeed, before one immerses oneself in a problem, one must be willing to engage in the problem and persist in the engagement.

Along with the historical development of a mathematical field, corresponding mathematics curricula are transformed through processes of didactical transposition (Brousseau, 1997), often mirroring, in part, the same societal needs. Curricula too can be organized around LoGINs, albeit necessarily different from the LoGINs of the corresponding scientific fields. Since the formation of school curricula is influenced by social and cultural factors, their LoGINs might differ across countries and even across regions withing the same country. However, for a curriculum to be a LoGIN-based, its instructional activities must be organized around investigations aimed at intellectually necessitating concepts and ideas that gradually lead up to the resolution of the LoGIN problems. These investigations must progress along trajectories of local intellectual needs: problematic situations that necessitate particular concepts and ideas.

Are current calculus curricula structured along LoGINs in this sense of the term? I did not conduct a thorough analysis to answer this question. However, based on my knowledge of current calculus instruction (see, for example, Harel (2021)), I stipulate that the answer is negative. However, from a DNR perspective, mathematics curricula, in general, and calculus curricula, in particular, should be structured along a LoGIN. LoGINs are not unique; nor are their corresponding trajectories; for they are determined by various factors, including, but not limited to (a) instructional objectives; (b) conceptual frameworks, such as APOS (Dubinsky and McDonald, 2001), Three Worlds of Mathematics (Tall, 2004), Covariational Reasoning (Thompson and Carlson, 2017)[2], and action theories (i.e., informal approaches based on practical experience); and (c) nature of content (e.g., differential based versus limit based, rigorous versus intuitive, paper-and-pencil based versus computer technology based).

3.1. A calculus curriculum LoGIN

In this section, I outline the set of overarching problems constituting a calculus curriculum LoGIN, rooted in two elements: quantitative reasoning and computational fluency.

Quantitative reasoning: There is a consensus among mathematics educators that quantitative reasoning must be a central goal of mathematics and science education at all grade levels. In DNR terms, quantitative reasoning is a way of thinking, a habit of mind, defined by Thompson (1990) as “an individual’s analysis of a situation into a quantitative structure” (p. 13). Bressoud (2023) quotes Steen’s (2001) description of the distinctive nature of quantitative reasoning:

> … Despite its occasional use as an euphemism for statistics in school curricula, quantitative literacy is not the same as statistics. Neither is it the same as mathematics, nor is it (as some fear) watered-down mathematics. Quantitative literacy is more a habit of mind, an approach to problems that employs and enhances both statistics and mathematics. … often anchored in data derived from and attached to the empirical world. Surprisingly to some, this inextricable link to reality makes quantitative reasoning every bit as challenging and rigorous as mathematical reasoning. (p. 2)

[2] See Bressoud and Ghedamsi (2016) for a comprehensive survey on state-of-the-art of such conceptual frameworks.

Computational fluency. A complementary element of this LoGIN pertains to the historical and pedagogical role of the methodologies and the notational system of calculus. Courant (1959) describes succinctly this historical role:

> Among the limiting processes of analysis there are two which play an especially important part, not only because they arise in many different connexions, but chiefly because of the very close reciprocal relation between them. Isolated examples of these two limiting processes, differentiation and integration, were considered even in classical times, but it is the recognition of their complementary nature and the resulting development of a new and methodical mathematics procedure that marks the beginning of the real systematic differential and integral calculus. The credit for initiating this development belongs equally to the two great geniuses of the seventeenth century, Newton and Leibnitz …. (p. 76; emphasis added)

I take the phrase "more or less routine methods" to mean what Smith and Thompson (2007) refer to as school algebra: "expression, manipulation, and formalization of mathematical concepts and structures mediated by explicit, rule-governed notational systems". (p. 95). These "routine methods" had a major breakthrough in Vieta's development of symbolic algebra and played a monumental role defining modern mathematics (Klein, 1968). Problem 5 in the proposed LoGIN (below) intents to capture the role of computational skills in the learning and teaching of calculus—that conceptual understanding and computational fluency are two sides of the same coin. The instruction of the module modeled this DNR's definition of computational skills, in that the students learn (a) to be cognizant of the quantitative reality represented by algebraic expressions; (b) to recognize that algebraic expressions are manipulated not haphazardly but with the purpose of arriving at a desired form and maintaining certain properties of the expression invariant; and (c) that in critical stages through this process they are able to pause and examine the meaning and implication of the result obtained.

These two elements together with curricular and historical considerations led to the following a calculus curriculum LoGIN organized in three categories of inextricably linked problems.

Category 1

1. Accumulation Problem: Given the rate of change of quantity A with respect to quantity B, how to determine the accumulation value of A for each value of B? (e.g., given how fast a quantity is changing at every moment, how much of it there is at every moment?)

2. Rate of Change Problem: Given the accumulation value of a quantity A for each value of quantity B, how to determine the rate of change of the accumulation value of A for each value of B (e.g., given how much of a quantity there is at every moment, how fast it is changing at every moment?)

The area of study of the first problem is known as integral calculus, and that of the second as differential calculus. These two questions constitute the intellectual need for linearization and with it the Fundamental Theorem of Integral Calculus (FTIC).

Category 2

3. Characteristic Value Problem: Given a covariation between two quantities A and B, how to determine values of specific characteristics for A and B?

This problem constitutes intellectual needs for knowledge about the existence, uniqueness, or computation of values of particular characteristics. Examples include theorems on extreme values, theorems on the mean values for rate of change and accumulation and theorems on differential equations.

4. Approximation Problem: Given a covariation of two quantities A and B, how to approximate values of A or B? In particular, how to approximate these values using elementary covariations formed by a finite number of rational operations? This problem constitutes intellectual needs for the knowledge about approximation methods, such as Newton Method, numerical integration, and infinite series, as well intellectual needs for the central theorems on power series.

Category 3

5. Methodical Problem: What methodologies governed by notational systems facilitate productive investigations of these problems? This problem reflects the intellectual need that culminated in the work of Newton and Leibniz, who are "responsible for developing the ideas of integral calculus to the point where hitherto insurmountable problems could be solved by more or less routine methods." (Apostol, 1967, p. 156).

Again, my stipulation that the core of undergraduate calculus curricula can be interpreted in terms of a LoGIN constituted by these problems does not imply that current calculus curricula in the U.S. are structured as such. Rather, the claim is that seen through a DNR lens, the curricular content of undergraduate calculus can be—but it does not seem to be—organized around a LoGIN comprised of these problems. While one can point to the presence of elements of these problems in current calculus curricula, one can hardly detect trajectories comprised of series of perturbation-resolution based instructional activities organized around investigations aimed at intellectually necessitating concepts and ideas that gradually lead up to resolutions of LoGIN problems.

Consider the following two episodes pertaining to the Fundamental Theorem of Integral Calculus:

Episode 1: A dialogue with a high-school student taking a calculus class.

Researcher: How was school today?
Student: Fine.
Researcher: How was your calculus class?
Student: Strange …
Researcher: Strange? What was strange about it?
Student: We learned a useless theorem.
Researcher: A useless theorem? Which theorem did you find useless?
Student: Not me—the teacher. … He thinks it is the most useless theorem in calculus.
Researcher: What was it?
Student: I don't remember, let me get my notebook. [Student gets her notebook and started flipping through the pages …] It is called … [reading slowly with pauses between words] the fundamental theorem of Integral calculus …
Researcher: Did the teacher say that? … Perhaps he was just joking …
Student: No, he was not! He was serious!
Researcher: Why is the theorem labeled fundamental then?
Student: I don't know. I wanted to ask him that, but I was scared. …

Episode 2: Undergraduate students' confusion about the Fundamental Theorem of Integral Calculus. The episode occurred in a recitation session conducted by a teaching assistant (TA). The TA solved the following problem: A population of bacteria living in a Petri dish has been treated with an experimental drug. If there are 10^6 bacteria initially and the rate of the growth at time t hours is observed to be $r(t) = 10^5(1 + t^2)$, find the total number of bacteria in the Petri dish after 3 hours.

TA: We could not solve this problem without the Fundamental Theorem of Integral Calculus, so we see just how important this Theorem is. …
Student A: I'm really confused about what you just said. In fact, I don't get the whole Fundamental Theorem thing. I thought the integral was the antiderivative, and now we are integrating the growth rate to get the population. I don't get it …
Student B: … and what does all this have to do with the derivative of $\int_a^x f(t)dt$? I don't understand that part at all.

Why would a mathematics teacher think of the Fundamental Theorem of Integral Calculus to be useless? And why does the confusion about the theorem continue to persist even in undergraduate calculus? To answer this question, one only needs to examine how this theorem is developed in textbooks. Typically, the development is comprised of four stages along the following lines.

Stage 1: Computing area through Riemann's Sum: $R(f,P,C)$, where P is the partition of the interval $[a, b]$, i.e a division into subintervals $[x_i, x_{i+1}[$, and C is the set of c_i chosen in each subinterval ($R(f,P,C)=\sum f(c_i)(x_{i+1}-x_i)$).

Stage 2: Definite integral

$\int_a^b f(x)dx = \lim_{\|P\| \to 0} R(f, P, C)$

Definite integral as a signed area

Properties of the definite integral

Stage 3: Indefinite integral: $\int f(x)dx = F(x) + C$ means $F'(x)=f(x)$

Stage 4: The Fundamental Theorem of Integral Calculus

Part 1: Assume f is continuous on $[a,b]$, if F is an antiderivative of f on$[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Part 2: Assume f is continuous on an open interval (a,b), then the area function $A(x) = \int_a^x f(t)dt$ is the antiderivative of f on I; that is, $A'(x)=f(x)$.

Bressoud (2011, p. 99) captures the didactical obstacle that commonly results from this presentation.

The common interpretation [by students and teachers, inferred from Stage 3] is that integration and differentiation are inverse processes. That is fine as far as it goes. The problem arises from the fact that this theorem assumes that the definite integral has been defined as a limit of Riemann sums. For most students, the working definition of the definite integral is the difference of the values of "the" antiderivative [inferred from juxtaposition of Stages 3, 4a and 4b]. When this interpretation of the theorem is combined with the common definition of integration, this theorem ceases to have any meaning [as these two episodes demonstrate].

Observations such as these, together with years of experience facing the difficulties students have with calculus concepts led to the development of several modules designed exclusively for prospective secondary teachers. One of these modules is discussed in the next section[3] .

4. Breaking the impasse: Reconceptualization through calculus re-education

Some of the articles in the ZDM special issue, Calculus in High School and College Around the World (Thompson and Harel, 2001), describe the efforts to implement research on student learning in redesigning national calculus curricula around the world. The guest editors reflect on these efforts through several research questions. One of these questions alludes to teachers' understandings of content, cognition, and pedagogy pertaining to calculus education. As was discussed earlier, the problem in

[3] Four such modules in geometry, calculus, conic sections, and problem solving, were developed.

the U.S. is that teachers take the “mainstream” calculus sequence offered in most universities and teach the same course upon their induction in high school. Reforming the “mainstream” calculus course turned out to be institutionally very difficult. To break this impasse, I developed, with the help of three colleagues in my institution, a calculus module for prospective teachers who have already taken the “mainstream” calculus sequence.

Twenty-five students took this module—all were prospective secondary teachers in their junior or senior year. The module lasted ten weeks, with three 50-minute lessons each week and was conducted as an exploratory teaching experiment (Steffe and Thompson, 2000). It was taught by me and observed by two of my colleagues, Professors Stevens and Rabin, and Mr. Fuller, a graduate student at the time. The events in each lesson (presentations, discussions, students’ questions and responses) were recorded by Professor Stevens and collaboratively edited by the four of us for accuracy of record. The students often worked on problems in small-working groups, discussed their solutions and approaches in whole-class settings, and, with prompts from me, identified the relevance of what they have learned to what might teach in the future. In this respect, this was a pedagogical-mathematical module focusing mainly on the first and fifth problems in the above outlined LoGIN.

The module’s instructional treatment consisted of a sequence of perturbation-resolution pairs rooted in the need to (a) model quantitative realities and (b) solve rate-of-change-accumulation problems as they arise in such realities. Accordingly, the choice of the problems, their structures, and their formulation was guided by the goal of creating among students the intellectual need to interpret problematic situations quantitatively in terms of functions and rate of change and accumulation. The next two subsections discuss an illustrative sample of this treatment and provide a portrait of the overall intellectual environment in the classroom.

4.1. Function, rate of change, and accumulation

The module commences with problems where there is a need to construct a function to model a natural phenomenon. The data that is typically available to us consists of how the phenomenon changes. Indeed, one of the main purposes of examining rates of change is to use information about rate to gain information about a function, a purpose which is often masked in traditional calculus courses. This, in turn, necessitates an in-depth study of rates of change, in particular, an exploration of average rate of change which leads gradually to an intuitive notion of instantaneous rate of change. To illustrate, consider the following two examples of the first set of problems discussed in class.

Problem 1a. A spherical balloon is expanding. You want to determine the volume of the balloon at any given instant from the moment it started to expand. What do you do? Students were asked to discuss the question in their small working group. Notice the standard context of this open-ended problem, on the one hand, and its unusual presentation, on the other hand.

Perturbation. Initially students get perplexed by this item because of its open-ended nature.

Resolution. After some discussion in their small working groups, they raise the need to obtain empirical data of the volume at a sequence time intervals.

Perturbation. The students are presented with the second item.

Problem 1b. Suppose that the balloon has already been partly blown up, and the time it took the radius to grow each half centimeter was taken and recorded below. To simplify notation, we use the radidus r instead of Δr, assuming $r=0$ at $t=0$.

Time in seconds	Radius in cm
0.00	0
0.06	0.5
0.50	1
1.69	1.5
4.00	2
7.81	2.5
13.50	3
21.44	3.5
32.00	4
45.56	4.5
62.50	5

The way in which the data is presented is critical. The table is built assuming that the volume is growing linearly, so that the radius is proportional to the cube root of time; specifically, $t(r) \approx 0.5r^3$. It is significant here that students conceive of time as a function of radius, opposite to the usual presentation. Once they conceive of this, they may think of taking successive differences, in which case they find that the third differences are essentially constant. The fact that the given values for radius are not all integers makes it difficult to guess a cubic without finding differences. Thus, students examine the rate of change of time with respect to radius. Moreover, the independent variable, namely the domain of the function $t(r)$, appears on the right in the table, so that students do not “learn” to always look at differences in the rightmost column.

Resolution: Typically, students provide the following approaches to solving Problem 1b, each of which is explored mathematically and pedagogically in whole class discussions.

Choosing "easy" values of *t* such as 0, 0.5, 4, 32, which led to $t(r)=0.5r^3$.

Using the observation that the third differences of the sequence of times being approximately all equal to 0.38, to construct a linear system of four equations out of a cubic function using the formula $V(r(t)) = \frac{4\pi}{3}\left(r_0 + \frac{\Delta r}{\Delta t}t\right)^3$.

Constructing explicitly the equation $\frac{\Delta V}{\Delta t} = \frac{4\pi}{3}\left(\frac{{r_i}^3 - {r_j}^3}{t_i - t_j}\right)$ for pairs of data (t_i, r_i).

Problem 2a. A cylindrical storage tank is full of uranium hexafluoride (UF6). You want to find the mass of the gas. What do you do?

As in Problem 1, students are asked to discuss this item in their small working group. Typically, the classroom interactions evolve through the following perturbation-resolution phases.

Perturbation: Students are initially quite confused by the first part of the problem despite the experience they had with the first problem.

Resolution: The instructor responds by drawing a cylinder on the board. Given the visual representation, some students suggest finding the volume through the formula $V=\pi R^2 H$ and sampling the density *D* and reminded the class of the formula $M=VD$, where *M* is the mass.

Perturbation: The instructor responded by asking the students if we were done, i.e. if the problem was solved. Then one student asked, "Are we assuming that density is uniform?".

Resolution: Another student responded by noting that the density will be higher at the bottom of the tank since the pressure is higher there. Thus, the class collectively concluded that density is not uniform. Some students suggest measuring the density at different levels.

Perturbation: The instructor presented the data for the second part of the problem.

Problem 2b. Suppose the cylinder has a radius of 3m and a height of 100m. When you sample the density at 10m increments of height, you find the following:

Height in meters	Mass in mg/cm^3
0	16.09
10	15.87
20	15.65
30	15.43
40	15.22
50	15.01
60	14.80
70	14.60
80	14.40
90	14.20
100	14.00

Resolution: The following are various responses typically provided by students.

Most students perceived that the data they were given followed a piecewise linear pattern (actually the density decreases exponentially), and they found an expression for density (sometimes labeled mass) as a function of height.

Some students used a Riemann sum (not necessarily labeled as such) to estimate the overall mass. For instance, one student used the function for density to find the density at centimeter increments and multiplied that by each piece of volume (using a spreadsheet to perform the actual calculations).

Other students attempted to use calculus to find the exact mass. They graphed density as a function of height from 0 to 100 meters and recognized that they were looking for the area under this curve.

Some set up an integral (not always correct) and solved.

Others recognized that the area could be found directly by using area formulas for familiar polygons (triangles, rectangles, trapezoids). Once this area is found, it can be multiplied by the constant πR^2 to give mass. Unit conversion presented difficulty for most students; they were uncertain what units their answer would be in or how to move between different units. [It was intentional that the density data is given in mg/cm^3, which is equivalent to kg/m^3, so that students would have to confront the issue of units and figure out how to resolve it]. The common strategy for resolving units was to express all measurements in centimeters instead of meters, yielding an answer in milligrams.

Each lesson ends with students reflecting on what they have learned, both mathematically and pedagogically. For example, a lesson pertaining to these problems is that often the need to describe a physical situation involving accumulation is resolved through studying in depth the models (i.e., functions) constructed from data about rate of change; this, in turn, facilitate an estimation of accumulation.

4.2. The fundamental theorem of integral calculus

A sequence of quantitatively based problems comprising the trajectory that leads up to the Fundamental Theorem of Integral Calculus. The trajectory consists of six phases, which collectively form three (instead of the two traditional) parts of the Fundamental Theorem of Integral Calculus.

Perturbation: Given a rate of change function $\frac{d}{dx}f$ over an interval *[a, b]*, how do we compute the net change *f(a) – f(b)?*

Resolution:

Definition of Riemann's Sum, R_N.

Definition of Integral: $\int_a^b \frac{d}{dx} f dx = \lim_{n\to\infty} R_N$ (when the limit exists)

Perturbation: Must $\int_a^b \frac{d}{dx} f dx = f(b) - f(a)$?

Resolution: Yes, if $\frac{d}{dx}f$ is integrable over *[a, b]* (i.e., if $\lim_{n\to\infty} R_N$ exists): That is:

FTIC (Part 1.1): If $\frac{d}{dx}f$ is integrable over *[a, b]*, then $\int_a^b \frac{d}{dx} f dx = f(b) - f(a)$

Perturbation: So, any time we are to compute the net change, *f(a) – f(b)* we must first verify that $\frac{d}{dx}f$ is integrable over *[a, b]?*

Resolution: No, there is no such a need if $\frac{d}{dx}f$ is continuous. That is:

FTIC (Part 1.2): If $\frac{d}{dx}f$ is continuous, then it is integrable.

Perturbation: Phases 1 and 2 imply that to find the net change of a rate of change function, there is a need to compute an antiderivative of that function.

Resolution: Indeed so. For this reason, we learn techniques of integration.

Perturbation: How do we compute accumulation (i.e., net change) of a covariation f that does not lend itself naturally or easily to rate of change, such as area, volume, and work?

Resolution: Any net change can be computed through the antiderivative of rate of change. That is:

FTIC (Part 2): If f is continuous on $[a, b]$, then $\forall x \in [a, b]$ the accumulation $\int_a^x f(t)dt$ is a function whose derivative is $f(t)$, i.e., $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

The reader surely noticed how this learning trajectory of the Fundamental Theorem of Integral Calculus is different from the traditional presentation of the theorem. As can be seen in Phases 1 and 2, similar to Thompson's approach, rate of change and accumulation are inextricably linked from the start: accumulation is defined in terms of rate of change! Students in the module are given ample opportunities to reason repeatedly through quantitatively based problems so that they (a) internalize the idea that accumulation is computed or approximated through Riemann's sum of a rate of change function; and (b) that the definite integral is merely a symbolic representation of the limit of Riemann's sum. In turn, this experience generates with the students the need to develop techniques of computing antiderivative (i.e., techniques of integration; Phase 4). As to Phases 2 and 3, the mathematical level of the instructional treatment depends on the rigor level of the course.

There remains Phase 6. Typically, the equation $\frac{d}{dx}\int_a^x f(t)dt \ = f(x)$ is presented as the first part of the Fundamental Theorem of Integral Calculus. Its intention, which is hardly present in students' conceptualization, is to indicate that "we can use [the limit $\lim_{n\to\infty} R_N$] to produce an antiderivative for any continuous function" (Bressoud, 2011; p. 112). This is an important fact which is strongly emphasized in the module. In this trajectory it is expressed in terms of a quantitative reasoning need, because the question whether each function lends itself to a rate of change interpretation was raised by students as well as by calculus instructors.

An unavoidable difficulty in teaching this module is that the population of students is familiar with the mechanics of calculus, i.e. formulas for differentiation, strategies for solving related rates problems, etc., even though most of these methods are void of meaning for them. Thus, it is expected that students will apply such methods to solve assigned problems. The strategy employed in the module is not to deter the students from using these methods, but to institutionalize a didactical contract that the class as a whole work to unpack these methods with questions such as "Do we know why the method used work?", "Are there are other ways to solve the problem?", "What are the cognitive benefits of various ways to student learning?", etc.

5. Summary and further research questions

The paper focuses on three approaches to sequencing of calculus content: the DI approach, the ID approach, and the TI approach, addressing three central questions: 1) Which should be taught first, differentiation or integration, or should the two be inseparable? 2) Does order impact student learning? 3) What is the relative impact of the three approaches on calculus learning as students transition from high-school mathematics to tertiary mathematics?

The paper situates the third question in the context of the continuity problem viewed from the perspective of Brousseau's theory of didactical versus epistemological obstacles. It further instantiates the question in the context of the transition from high school calculus to university calculus in the U.S. The paper points to a major didactical obstacle inherent to the content and structure of calculus courses offered by universities and high schools; namely, that the uniformity in content and low rigor across the two educational levels result in a "smooth" transition from high school to university. The paper examines a series of nine factors constituting the claims in favor of either the ID approach or the DI approach. Based on an analysis of these factors and their respective claims, the paper offers the observation, initially made by Thompson, et al. (2013), that the ID and DI approaches might—perhaps inadvertently—promote the view among students that calculus is comprised of two parts that exist independently but are tied together with the Fundamental Theorem of Integral Calculus. Against this view stands the TI-based approach, which purposely promotes the alternative view that rate of change (differentiation) and accumulation (integration) as inseparable.

The paper concludes with an outline a project that tackled the continuity problem through a module aimed at elevating the quantitative rigor of calculus among prospective secondary mathematics teachers through reflection and reconceptualization of what they have learned in their first "mainstream" calculus courses. The design of the module and its instructional implementation were modeled through the principles of DNR-based instruction in mathematics. The module was organized around investigations aimed at intellectually necessitating concepts and ideas that gradually lead up to better understanding of two of the five problems constituting calculus' locus of global intellectual need (LoGIN), the accumulation problem and the rate of change problem, with less explicit emphasis due to time constraints, on the other three LoGIN problems: the characteristic value problem, the approximation problem, and the methodical problem.

In conclusion, I list additional research questions entailed from the analyses discussed in this paper:

1) What impact does a module such the one discussed in this paper have on the participants' teaching practices of mathematics, in general, and calculus, in particular? For example, what role would quantitative reasoning play in their instruction as they get inducted in schools?
2) The population of the module was familiar with the calculus concepts addressed in the module. Even if their familiarity was largely procedural, there likely to be competing conceptualizations between those formed in the "mainstream" calculus courses and those promoted in the module. What impact does this fact have on their long term retention and conceptualization of calculus concepts?
3) The idea of designing and implementing calculus courses along a locus of global intellectual need (LoGIN) was only partially implemented in the module discussed in this paper. To what extent are LoGIN-based calculus

courses viable given the social and institutional environment of calculus education in the U.S.?

4) Would modules for teachers, such as the one outlined in this paper, further common understanding between high-school teachers and university instructors as to (a) what calculus should be taught? and (b) how should that calculus be taught?
5) What epistemological obstacles and didactical obstacles associated with the DI approach and the ID approach?

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